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Letter

New sum rules of the Koide type

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ABSTRACT

We report a mass rule of Koide type with inverse shape,

$m_i = M^{(d)}(w_0 + w_i)^{-2}$.

It applies to the down-quark sector with numerical precision comparable to that of the direct charged-lepton sum rule $m_i = M^{(l)}(z_0 + z_i)^2$. For central mass values, Koide ratio reaches exactly 2/3 near $Q \simeq 280$ TeV under Standard Model renormalisation-group running. We also review other rules of the direct kind involving quarks.

1. Introduction

The preon-program era, including its attempts to account for family textures and CKM angles, produced a number of striking mass relations. Among these, one of the most notable survivors is the Koide formula for charged leptons [1]

$$m_i = M^{(l)}(z_0 + z_i)^2, \ \sum_{i=1}^{3} z_i = 0, \ \frac{1}{3}\sum_{i=1}^{3} z_i^2 = z_0^2 \tag{1}$$

that is exact for lepton pole masses: given as input the masses of electron and muon, it predicts the tau mass within experimental precision. The quantities z_i were assumed to be the abelian charge of some preon. We refer to the two restrictions as “averaging constraints”, but it could also be trace conditions on some matrix realization.

Once the preonic justification disappears, it is not easy to find a reason for a Yukawa coupling going as a square, not to say as an additive charge. It is possible to accommodate the relation through a purpose-built Higgs potential, at the cost of introducing non-standard anomalous dimensions for the Yukawa operators [2]. But the formula anyway was thought to fail for quarks. This is not quite so; besides the known mixed tuples we will discuss later, the main goal of this paper is to give notice of the discovery of an inverse tuple for the down quarks

$$m_i^{(d)} = M^{(d)}/(w_0 + w_i)^2, \ \sum w_i = 0, \ \frac{1}{3}\sum w_i^2 = w_0^2 \tag{2}$$

that is exact at some point above 100 TeV and remains within one standard deviation of the experimental central values throughout the running. In some sense this is the opposite exactness of the lepton tuple, that keeps stably worse under running but works fine in the infrared.

2. The ratio form

It is conventional to rewrite the Koide relation in a form that eliminates the overall mass scale M. This practice has the advantage of condensing (1) into a single equation and the disadvantage of hiding the original additive charges that motivate the expression. With this format we would write

$$K(e,\mu,\tau) = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{1}{3}\left(1 + \frac{\langle z_i^2 \rangle}{z_0^2}\right) = \frac{2}{3} \tag{3}$$

for the charged leptons, and

$$K^{-1}(d,s,b) = \frac{1/m_d + 1/m_s + 1/m_b}{\left(1/\sqrt{m_d} + 1/\sqrt{m_s} + 1/\sqrt{m_b}\right)^2} = \frac{2}{3} \tag{4}$$

for the down quarks.

This form is well-defined, provided all charges are positive, as seems to be the case for both tuples.

The averaging constraint $\langle z_i^2 \rangle = z_0^2$ (or respectively with w charges in the inverse formula), $i \in \{1,2,3\}$, is equivalent to the ratio being $2/3$. With only positive charges, the left-hand side is bounded between $1/3$ and 1.

In some cases, instead of operating with the square root of each mass, it can be preferable to define explicitly quantities $q_i = M^{1/2}(z_0 + z_i)$, where M is the overall mass scale of the tuple. The connection is then $q_i^2 = m_i$, and one must take into account that the same particle mass can be represented by charges of opposite sign in different tuples.

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3. Other quark tuples

It has been conjectured that the whole of the quark sector could be covered by generation-alternating tuples

$$(tbc) \qquad (bcs) \qquad (csu) \qquad (sud) \tag{5}$$

that would correspond to four of the eight possible choices in

$$(ts) \times (ub) \times (cd) \tag{6}$$

Neither tuple fares as well as Eqs. (3), (4); the tuples (tbc) and (bcs) are within some percent in the infrared, especially when one chooses a "decoupling" approach where each mass is taken as $m(m)$ in some running scheme. The other two tuples fail in an interesting way: they work under the replacements $u \to 0$ and $d \to (u+d)$ much like in ChPT approaches.

Instead of the common scale analysis we used in the inverse down tuple, here we base the observation of these tuples in PDG 2025 values [3] in $\overline{MS}$ scheme: $m_s(2\,\text{GeV}) = 93.5 \pm 0.8$, $m_c(m_c) = 1273.0 \pm 4.6$, $m_b(m_b) = 4183 \pm 7$ MeV, $m_t(m_t) = 162.69\,\text{GeV}$, as we have found the puzzling fact that they hold better than at a common scale, and we wonder if it is related to some process of decoupling. With this caveat we have the following table, where K denotes the Koide ratio $\sum q_i^2 / (\sum q_i)^2$ and we mark with a minus sign the use of the negative square root for a charge, i.e. "-s" indicates the use of $q_s = -\sqrt{m_s}$

Tuple	K	Dev from 2/3
(t, b, c)	0.662729	−0.591%
$(-s, c, b)$	0.674802	+1.220%
(c, s, u)	0.624058	−6.391%
$(c, s, 0)$,	0.664478	−0.328%
(s, u, d)	0.565959	−15.106%
$(s, 0, u+d)$	0.664831	−0.275%

In the table, 0 and $u+d$ signal the above replacements, so that the charge associated to $u+d$ is $\sqrt{m_u + m_d}$ and the whole tuple $(s, 0, u+d)$ can be calculated from the quotient $m_s/\bar{m}_{ud} = 27.33$ [3], using then $\sqrt{2\bar{m}_{ud}}$ as the charge.

For comparison, we show also the no-replaced tuples (c, s, u) and (s, u, d) choosing as reference values for u and d the ones of [3, Sec. 60.5.1], namely 2.20 MeV and 4.69 MeV

The substitution $u \to 0$ is interesting both algebraically and historically. The first mention of a Koide-like tuple is the sdu set used by Harari et al. [4], substituting $u \to 0$ in the lower mass quarks as part of another scheme, based only in symmetry, to produce the Cabibbo angle. As for the other tuples, the possibility of fitting (t, b, c) was first mentioned by Rodejohann and Zhang [5, v1], and the tuple (b, c, s) is also well known [6], with the detail of being the only one that requires a negative charge, conventionally assigned to the s quark.

The exact algebraic chain, starting from an exact seed $(0, m_s, m_c)$ instead of experimental values, and generating new elements via the equation, is an interesting object by itself. For instance, the inverse tuple also "sees" the top quark. More precisely, the inverse-seesaw solution satisfies

$$m_d \cdot \sigma\left(\frac{m_t}{m_c}\right) = m_b, \tag{7}$$

where σ denotes the Galois conjugation $\sqrt{3} \mapsto -\sqrt{3}$. When going across the field algebra, we find the numbers $(151, 28)$ appearing both in $\frac{m_t}{m_c} = 151 - 28\sqrt{3}$ and also in the calculation of the inverse solution for m_d.

Considered as a whole, and adjoining the inverse tuple, it is amusing that a single formula provides enough mass relations to adjust the six quarks up to one overall scale. However, this observation should be interpreted cautiously: as we relax the precision of the mass rule, it is really not so complicated to find random masses fitting into them, as far as they are taken from a log-uniform distribution.

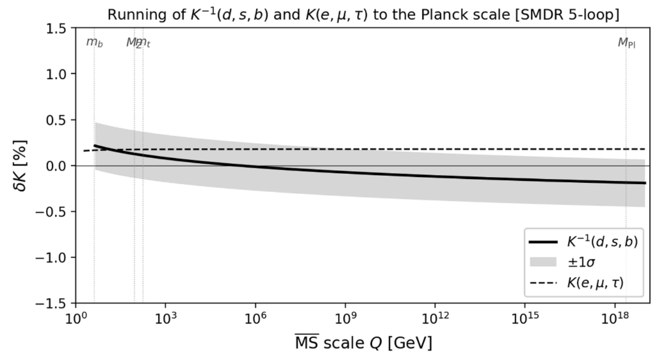

Fig. 1. RG running up to Planck mass.

4. Renormalisation group

Quotient tuples between particles of the same kind have the same exponents under the renormalisation group, and they are approximately invariant; the main source of perturbative change is the Yukawa coupling; that means that low-mass quark tuples such as (sud) or their variants run practically flat.

As for the inverse tuple of down-type quarks, for the desert hypothesis under renormalisation-group running (5-loop Standard Model, using the Martin–Robertson SMDR code [7]), we can see how the "inverse Koide ratio" improves toward the UV:

Scale Q	$K^{-1}(d, s, b)$	Deviation from 2/3
M_Z	0.66750	+0.13%
1 TeV	0.66719	+0.08%
100 TeV	0.66675	+0.01%
280 TeV	**0.66667**	**0.00**%
10^6 GeV	0.66657	−0.01%
M_{Planck}	0.66539	−0.19%

It crosses 2/3 at $Q = 280\,\text{TeV}$ using the default model of SMDR. It crosses 2/3 a bit earlier, at 10.18 TeV, if we interpolate the 2024-PDG Table 2 values from [8], which uses the REAP Mathematica code (permitting simultaneous CKM running) with Yukawa inputs at one digit less precision than our SMDR runs. As we see in Fig. 1, the equation is within one-sigma error along all the run, so more experimental precision is needed if we want to pinpoint the exact scale of a portal of new physics. The determination of the crossing point is very sensitive to variations in the mass of the down and strange quarks. A preliminary run with updated 2025 inputs shifts the crossing slightly towards lower energy but keeps the curve within the one-sigma band throughout the run, so the existence of a crossing is robust against current measurement updates

5. Other observations

It is interesting to examine the global mass scale of each tuple,

$$M z_0^2 = (m_1 + m_2 + m_3)/6 \tag{8}$$

The exact chain $(0, a, b) \leftrightarrow (-a, b, c)$ presents an obvious factor 3 between the corresponding $M z_0^2$ mass scales. This factor is observed between (e, μ, τ) and $(-s, c, b)$, but not in the quark chain. With PDG 2025 data as above, what we observe is a factor 2. More precisely, we get

$$z_0^{(-s,c,b)} / z_0^{(0,s,c)} = 1.99974 \tag{9}$$

with a deviation of 0.013% , still within 0.1σ of the experimental data, if we extract the z_0 from the square roots. On the other hand, if we extract the quotient from Eq. (8), the discrepancy with a factor 2 is greater, we get 2.01522. This can be interpreted as a hint that the charges, not their squares, are the main objects to establish sum rules here.

Another consequence of Eq. (9) is that it implies a sum rule between charges:

$$\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c} \tag{10}$$

Which alone predicts $m_b = 4184.5$ MeV vs experimental 4183 ± 7, a match to 0.04% that is noticeable.

While this match is striking, the relation belongs to a broader family $\sqrt{m_3} = (k+1)\sqrt{m_1} + (k-1)\sqrt{m_2}$, and its statistical significance relative to chance requires further study. Generically, relations between quotients $\sqrt{m_1/m_2}$ were deeply explored in 1980–1990 as source of textures for the CKM matrix [4,9–11].

6. Final discussion

The main conclusion of this note is that out of four equal charge tuples in the standard model, namely charged and neutral leptons, and up and down quarks, two of them are known to have mass sum rules of the Koide type up to one-sigma experimental precision. The neutrino tuple cannot be decided empirically (but see [12]), and the up-type quark seems to need an interleaving structure to match in this pattern. That the up-type quarks get a more complicate structure can be seen in models of flavour. As an example, in the susy-guided model of [13], when the flavour-family group is broken to a SU(3) family, the down type squarks remain in triplets but the up type squarks seem to need sextets.

That the shape of both sum rules differs by an inversion adds interest to the intuition, to be examined elsewhere, of obtaining Koide formula in scenarios of duality, such as the ones of Sannino [14] or more generic ones with explicit link between mesinos and leptons, as in [13].

A partial support for such Sannino or Seiberg-like duality scenarios is that it is easy to select actual mesons whose masses are near those of lepton and quark tuples, and so reproduce Koide formula in the meson sector. In some sense this exploits the proximity between the QCD scale and the (Yukawa-scaled) Fermi scale. Once the mass of pion is considered as a substitute of the μ lepton or the s quark then one can scan for either the pattern $(0, \pi, D)$ or the pattern (π, D, B), respectively.

There is no obvious clue of the mechanism that generates the mass as the square of an abelian charge. The initial mechanism of preons is discouraged by the existence of a sequence of tuples so that the same particle would need different decompositions in each tuple.

The possibility of chaining tuples to produce new masses is a mildly interesting mathematical problem involving sequences of involutions on the quadric equation,

$$x^2 + y^2 + z^2 = 4(xy + yz + zx) \tag{11}$$

while maintaining agnosticism about the dynamical model of the Koide equation. It supplements the approach of presenting each tuple as a rotation of the hypercharge component of a SU(3) symmetry.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used Claude (Anthropic) and GPT 5.4 (OpenAI) for algebraic exploration, numerical checks, and minor drafting assistance. The author reviewed and edited all the content and takes full responsibility for the publication.

The relation motivating this letter was noticed during an AI-assisted study for a different paper, namely on the use of Seiberg-like SUSY models to explain the lepton tuple. It became apparent after the accidental launch of a numerical exploration of inverse tuples by Claude Opus 4.6.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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