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LETTER

On Planck Areal Speed

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Abstract. We explore the concept of areal speed concerning minimal length scales associated with Planck units. In gravitational systems, the orbital radius measured in Planck units becomes independent of the Newton's constant; it is just a multiple of the Compton wavelength of M. Reversing the argument leads to an emergent understanding of Newtonian gravity.

We also note that the general relativity correction is compensated by a Generalized Uncertainty Principle and we highlight the critical role of spatial dimensionality.

1 Areal speed and minimal length

The question of the existence of a minimal scale of length associated with Planck units was proposed initially in the mid-sixties by [1], revisited by [2] in the eighties, and finally approached as a topic of general interest in the last thirty years (see [3–6] and review [7]). It happens via gravitational effects in measurement and usually it imposes extra terms in the indeterminacy principle. The most common example is the deformation into a Generalized Uncertainty Principle [8]

Usually, a minimal scale of length determines a minimal distance, a minimal area or a minimal time interval. We are intrigued by the case of the areal speed, defined as usual, the area per unit of time swept by an orbiting particle. For any particle of mass m we can build a natural areal speed $\dot{A}_m = \hbar/m = cL_m \sim v\lambda_m$, where L_m is the reduced Compton length of the particle and λ_m its De Broglie wavelength at tangential speed v.

Particularly we can build the Planck Areal Speed A_P from the Planck length L_P and the Planck time T_P

$$\dot{A}_P = \frac{L_P^2}{T_P} = cL_P = \frac{\hbar}{M_P} \tag{1}$$

And here an operational quantisation of the area and time involved in an areal speed does not quantise $\dot{A}$ itself. It becomes as much a fraction k of quantum numbers,

$$\dot{A} = \frac{p}{q}\dot{A}_P = k\dot{A}_P, \tag{2}$$

perhaps only of topological interest. Still, let's take some time to consider it.

In principle $\dot{A}_P$ is always smaller than most areal speeds happening in quantum mechanics. If an interaction has an orbit quantised by $J = n\hbar$, its areal speed is

$$\dot{A} = \frac{n\hbar}{2m} = \frac{1}{2}n\frac{M_P}{m}\dot{A}_P \tag{3}$$

and as long as the test mass m is well smaller than Planck mass, we can guarantee that the orbit areal speed is greater than Planck's.

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2 Kepler length

For a circular orbit with an inverse square force of coupling K, the radius of the orbit is

$$r = \frac{4m}{K}\dot{A}^2 = \frac{n^2}{Km} M_P^2 \dot{A}_P^2 \tag{1}$$

Now, something peculiar happens with gravity, where $K = GMm$. The radius has no dependency on the test mass, and Newton's constant cancels the Planck area:

$$r = \frac{4}{GM}\dot{A}^2 = 4L_M \frac{\dot{A}^2}{\dot{A}_P^2} = 4L_M \frac{p^2}{q^2} \tag{2}$$

so we get explicitly $L_M = \hbar/cM$, the reduced Compton wavelength of the particle creating the gravitational field. And *when the areal speed is one half of the Planck areal speed, the radius of the gravitational orbit around M is the reduced Compton length of the particle M.*

Of course this is a re-statement of

$$\dot{A} = \frac{1}{2}\sqrt{GMr} \tag{3}$$

And anyway this regime is outside of the expected range of areal velocities. It can be studied in the context of Compton-Schwarzchild duality [9, 10]

3 Gravity from lengths

A more interesting note, to us, is that we can reverse the argument to produce an emergent definition of Newtonian Gravity:

> A Newtonian gravity is a mutual force law with a single universal coupling constant between two arbitrary masses and such that for any mass M, the circular orbit of any test particle m at a distance equal to the Compton length of M has the same areal speed independent of M and m.

Let's try to prove the consistency of this definition, i.e., that the analytical form of a Newtonian gravity as defined above is

$$F(M, m, r) = G\frac{Mm}{r^2} \tag{1}$$

for some constant G

Let $\dot{A}_0$ be the universal areal speed at Compton length radius. We also define the constants $L_0 = \dot{A}_0/c$ and $M_0 = \hbar/\dot{A}_0$. For any test particle m, its angular momentum is

$$2m\dot{A}_0 = r_M m v, \tag{2}$$

where v is the tangential speed of the particle and $r_M = \hbar/cM$ is the Compton length radius of the orbit. The force in this situation is

$$F(M, m, r_M) = \frac{m}{r_M}v^2 = \frac{c\, mM}{\hbar}\frac{4A_0^2}{r_M^2}\left[\frac{cMr_M}{\hbar}\right]^n \tag{3}$$

For the force to be mutual, ie symmetrical under the exchange $m \leftrightarrow M$, we need $n = 0$. Now lets define

$$G_0 \equiv 4\frac{c\dot{A}_0^2}{\hbar} = 4\frac{\hbar c}{M_0^2} \tag{4}$$

and we see that, as expected, the functional form of the force law is, for any m and M -and thus for all radius r_M-

$$F(M, m, r_M) = G_0\frac{Mm}{r_M^2} \tag{5}$$

To see that this is the final closed form of the force equation, note that if a second force $K(M, m, r)$ meets the same conditions, the difference $\Delta \equiv F - K$ will be such that

$$0 = \Delta(M, m, r_M) = \Delta(m, M, r_M) = \Delta(m, M, r_m) \tag{6}$$

Thus it has two different zeros in the radius coordinate for each arbitrary pair (M, m) of masses, so any new solution needs another extra coupling constant to fit the other zero.

This proves the assertion, but depending in the condition of a single coupling constant. Note for example the family of general solutions

$$\frac{G_0 Mm}{I_0^{(2-n)}}\left(\frac{M^M}{m^m}\right)^{\frac{2-n}{M-m}}\left(\frac{M}{m}\right)^{-\frac{(2-n)Mm}{I_0(M-m)}r} r^{-n} \tag{7}$$

that have a secondary coupling I_0, a multiple of $\hbar/c$. A different attempt could be to ask that "for any I_0 defining a length for M, the areal speed at such length is the same, independent of M and m". In any case, for some applications, these postulates are surely weaker than the requirement of Bertrand's theorem.

4 Gravity, the strongest classical force

At Compton length from the center of force, all the forces would generate stable orbits a lot faster, in areal speed, than gravity. So if we had an independent way to define the minimal length, we would have another way to define gravity:

> Gravity is the strongest force that allows stable orbits with an areal speed as low as Planck's areal speed.

5 Extra dimensions

Which carries us to consider how the dimensionality of space affects the cancellation.

In general, if we allow force to be any power r^{-q} of the radius, the areal speed will be as

$$G^{\frac{1}{2}-\frac{1}{q}} m^{\frac{1}{2}} r^{\frac{3}{2}-\frac{q}{2}} c^{\frac{3}{q}-1} \hbar^{-\frac{1}{q}} \tag{1}$$

and only for $q = 2$ Newton's constant cancels out. Thus here we have a non-relativistic variant of the arguments of [11, 12] that justified the dimensionality of space-time from the properties of gravity.

In fact, our proof above fixes gravity to be $\sim 1/r^2$, similarly to [12]. But it could be objected that as we go to extra dimensions we have new independent angular momentum variables and they should be incorporated in the definition.

6 GUP and relativity

Finally, let us consider a first approximation to relativistic effects.

If we enable special relativity, the circular orbits of a given M, m system have a minimum possible angular momentum when the radius goes to zero, namely GMm/c. So in this regime the existence of closer circular orbits requires

$$\frac{GMm}{c} < 2m\dot{A} \tag{1}$$

and $2\dot{A}/c$ is bounded by the Schwarzschild radius, $r_M = GM/c^2$.

If we consider general relativity, the Schwarzschild effective potential amounts to a correction to the orbit radius that, being proportional to r_M, reintroduces the gravitational field. The perturbative modification of the Newtonian solution is

$$r = 2k^2 L_M \left[1 \pm \sqrt{1 - \frac{3L_P^2}{k^2 L_M^2}}\right] \tag{2}$$

and so

$$r_\pm \approx 2k^2 L_M \left[1 \pm (1 - \frac{3L_P^2}{2k^2 L_M^2})\right] \tag{3}$$

which introduces our solution of interest,

$$r_+ \approx 4k^2 L_M (1 - \frac{3}{4k^2}\frac{L_P^2}{L_M^2}) \tag{4}$$

as well as the branch of unstable solutions $r_- \approx +\frac{3}{2} r_M$

So we get an extra correction that seems to reintroduce Planck lenght L_P and thus Newton's constant. But we can also invoke the Generalised Uncertainty Principle. Because under GUP, the reduced Compton wavelength acquires an extra correction [13]:

$$L'_M = L_M (1 + \alpha \frac{L_P^2}{L_M^2}) \tag{5}$$

We can combine multiplicatively both corrections so that at first order they cancel, if $\alpha = \frac{3}{4k^2}$, and thus we still recover a pure radius, with the new "generalised reduced Compton length", that does not explicitly show a dependency of the Planck scale. In itself, this is an interesting result showing how general relativity and the generalised uncertainty relation interplay.

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