Some conjectures looking for a NCG theory

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Abstract

It is pointed out that ambiguities in the regularization of actions with second derivatives seem to happen with the same multiplicity that the standard model of elementary particles

1 Introduction, apology and apologetics

First of all, I must apologize by bringing up a such speculative paper as this is. Our justification is to open a forgot question which could fit in the current research for the fine structure of differential geometry [1]. While the anticommutativity of Cartan differentials has the obvious appeal for physicists, the traditionally troubled development of geometry has precluded us of getting to this point at the adequate moment: Cartan theory itself was solidified lot of time before we got to zero the research in a set of four elemental particles and some multiplicities and generations. And even if this "tetrad" of particles could have constituted a clue, we have waited another quarter of century to get a theory of deformed, non commutative, differential calculus able to contain it [3].

Cartan differentials are fundamental geometrical objects, they carry the modern interpretation of Poincare principle, a prerequisite to get the fundamental theorem of integration. A future theory based on such object would be epistemologically as fundamental as the curvature-based relativity theory. But it would be probably a fool attempt to try do derive such theory directly from the sky of mathematical principles. Instead of this, we must to try to stick to earth; our model should be as near to possible to the observed spectrum. In fact our initial inspiration comes from the more pragmatic approach to field theory, namely lattice QFT.

Lattice theorists usually find themselves fighting against unwanted degrees of freedom coming from (supposed) discretization artifact: fermion doubling, Gribov copies, FP doublers and so on. Such infection is supposed to disappear in the continuum limit. But on the other side, arguments as triviality, localization, gravity etc drive to consider the existence of a true natural cutoff, a minimum lattice length. It seems difficult to marry both views.

In this paper we stop confronting head-to-head against doubling and we sketch the opposed move: to look explicitly for them in the very core of mathematics, (differential) geometry. We shall find that, by deforming differential calculus, the four dimensional coordinate systems gets the number of degrees of freedom implied by the fermions of the standard model. So, we propose to use doubling as a tool to build fermionic actions following classical rules, as for instance Einstein-Hilbert.

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2 A very nice coordinate system is discretized

Our starting point is, as we have said, four dimensional (1,3) Riemannian geometry. The core of our proposal is to choose a coordinate system suitable for scattering experiences; specifically we choose to use Schwarzchild coordinates, t, ρ, θ, ϕ. The volume form would be then the usual wedging $dt d\rho d\theta d\phi$ of anticommuting differentials. The coordinates have the usual range: $t \in R, \rho \in R^+, \theta \in (0, \pi), \phi \in S^1$.

Note that this coordination has interesting peculiarities: It is singular for $\rho = 0$, then any calculation must avoid to cross this point. Also, only two differential terms carry length units. The angles θ, ϕ will get units as an effect of the discretization, but the continuum limit must be free of units, very much as an asymptotic freedom effect. The angle θ has also a singularity, but it is different from the one of ρ and we do not see the most adequate method to manage it. Perhaps we would to build a double cover of the space or to do some trick to define the coordinate over RP^1.

Having the coordinate system in mind, consider first derivatives on any coordinate x^i.

$$\frac{\partial A}{\partial x^i}(x) \to \frac{A(\phi_i^+(x, \epsilon)) - A(\phi_i^-(x, \epsilon))}{\Delta \phi_i} \tag{1}$$

The relationship between numerator and denominator, given by $\Delta\phi_i = \phi_i^+(x, \epsilon) - \phi_i^-(x, \epsilon)$, could break when the theory is quantized. Any term containing a derivative of A can thus be replaced by a term depending of A^+, A^-, and a scale m_i^{-1}.

If the coordinate ρ is involved, we have an additional restriction, as we can not cross $\rho = 0$, then we must impose a positivity condition, $\phi_\rho^\pm(x^\rho) > x^\rho$.

To resume, if we want to calculate and action involving first derivatives in a four dimensional Riemannian space, we must work with a duplicated coordinate system, where one of the coordinates has a restriction to its duplication, and two coordinates must to be free of length units. Neutrino and quarks come fast to the mind.

Ideally, we could try to formalize the doubling by rewriting actions in terms of (real,complex?) fermions.

3 An action with second derivatives

Suppose that we have some term in the action needing to evaluate a second derivative. The same argument doubling fields above, will then drive us to triple the number of fields here, as we need to specify three points for each calculation. But this is not bad: It only implies that such actions must be discretized by using three "generations" of coordinate functions.

$$\frac{\frac{A(\phi^r)-A(\phi^c)}{\phi^r-\phi^c} - \frac{A(\phi^c)-A(\phi^l)}{\phi^c-\phi^l}}{\phi^+ - \phi^-} \tag{2}$$

Again, the singularity at $\rho = 0$ imposes a restriction to the triplication of the radial component, but is is difficult to trace how this restriction affects to the whole structure. Mixing and CKM matrix come to mind, if only as metaphor (But mixing seems to be relevant even without restrictions, see section 5).

It would be examined what to do when a Lagrangian contains both first and second derivatives. To tensor both freedoms seems to be the more reasonable approach, as it clones the known particle spectrum. But to build the second derivatives by composing the freedom got in the first ones is also a reasonable assumption, very much as to use a technicolor technique to get the generations going.

This latter approach is similar to compose two spin one-half representations to get one of spin one. But it is interesting to note that in our case we have an additional discrete ambiguity, namely to reverse the order of the differences, as its composition counterweights the change of sign.

Anyway the moral is that, somehow, when trying to build an higher derivative action, as for instance gravity, our coordinate system is forced to approach the shape of the experimentally known fermions.

This technique is not exclusive of gravity; it could be applied in principle to any higher derivative model, for instance [9]. It is interesting to note that some higher derivative models have been related to antiferromagnetic fixed points.

Just for enlightenment, lets contemplate one expanded form of Einstein-Hilbert action

$$
\begin{aligned}
S \;=\int(\quad & -g^{lq}g^{in}\frac{\partial^2 g_{nl}}{\partial x^i \partial x^q} + g^{lq}g^{in}\frac{\partial^2 g_{ql}}{\partial x^i \partial x^n} \\
& -g^{lq}\frac{\partial g^{in}}{\partial x^i}\frac{\partial g_{nl}}{\partial x^q} + g^{lq}\frac{1}{2}\frac{\partial g^{in}}{\partial x^i}\frac{\partial g_{ql}}{\partial x^n} + g^{lq}\frac{1}{2}\frac{\partial g^{in}}{\partial x^l}\frac{\partial g_{ni}}{\partial x^q} \\
& -g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{ni}}{\partial x^p}g^{pm}\frac{\partial g_{ml}}{\partial x^q} - g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{ni}}{\partial x^p}g^{pm}\frac{\partial g_{qm}}{\partial x^l} + g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{ni}}{\partial x^p}g^{pm}\frac{\partial g_{ql}}{\partial x^m} \\
& -g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pn}}{\partial x^i}g^{pm}\frac{\partial g_{ml}}{\partial x^q} - g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pn}}{\partial x^i}g^{pm}\frac{\partial g_{qm}}{\partial x^l} + g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pn}}{\partial x^i}g^{pm}\frac{\partial g_{ql}}{\partial x^m} \\
& +g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pi}}{\partial x^n}g^{pm}\frac{\partial g_{ml}}{\partial x^q} + g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pi}}{\partial x^n}g^{pm}\frac{\partial g_{qm}}{\partial x^l} - g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pi}}{\partial x^n}g^{pm}\frac{\partial g_{ql}}{\partial x^m} \\
& +g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{nl}}{\partial x^p}g^{pm}\frac{\partial g_{mi}}{\partial x^q} + g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{nl}}{\partial x^p}g^{pm}\frac{\partial g_{qm}}{\partial x^i} - g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{nl}}{\partial x^p}g^{pm}\frac{\partial g_{qi}}{\partial x^m} \\
& +g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pn}}{\partial x^l}g^{pm}\frac{\partial g_{mi}}{\partial x^q} + g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pn}}{\partial x^l}g^{pm}\frac{\partial g_{qm}}{\partial x^i} - g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pn}}{\partial x^l}g^{pm}\frac{\partial g_{qi}}{\partial x^m} \\
& -g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pl}}{\partial x^n}g^{pm}\frac{\partial g_{mi}}{\partial x^q} - g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pl}}{\partial x^n}g^{pm}\frac{\partial g_{qm}}{\partial x^i} + g^{lq}\frac{1}{4}g^{in}\frac{\partial g_{pl}}{\partial x^n}g^{pm}\frac{\partial g_{qi}}{\partial x^m})d\Omega
\end{aligned}
$$

(It is feasible to factorize the metric in a product of vierbeins, and then to make the gauge explicit, but we will refrain from doing it here)

Note that the number of different terms in the gravity action can be classified in three kinds, very much as the gauge fields of the standard model can be ordered inside elements of a algebra of three components, $C \oplus H \oplus M_3(C)$. Again, this points to consider NCG (but please do not take too seriously this argument, it is only a remark out of curiosity).

Either we take the metric or its component vierbein as the field to minimize, we confront an action with second derivatives. Note also that first derivatives appear always in multiplicative pairs, which seems difficult to fit with the fermionic action, and that second derivatives contain crossed terms, which the standard model could be controlling through coupling constants, instead of masses.

4 Shadows of NCG, the Higgs and Masses

Once we have accepted the discretization of derivations, we find us with a big collection of mass like terms, the mass being related to the size of our lattices, i.e., to the denominator of each derivative. This result is not nice, as mass must be generated through couplings with the Higgs.

In some sense, the Higgs should control the discretization. As the mass of the Higgs goes to infinity, so must go the fermionic ones. In the high energy, short distances, regime, every mass is negligible, and we would expect to be performing a fully non commutative calculus. *In the low energy, large distances, regime, masses can be taken to be infinite, then its corresponding "distances" approach zero, and we are taking usual derivatives*, thus approaching a traditional commutative action.

Also, this effect relates to asymptotic freedom, as only at low energy our coordinate system needs to make unobservable the angular variables, while at high energy they can perform as free particles.

Our puzzle lacks of two pieces: one natural geometrical interpretation of Higgs mass, and a justification for not going to the infinitesimal limit. Non commutative geometry models provide the first piece, by relating the Higgs mass with a separation between multiple sheets of space-time. The above quoted antiferromagnetic models could be an alternative if we want to remain in the statistical world of lattice QFT.

To see how NCG is able to hold so complicated interrelations, it is illustrative to make a one dimensional digression: Take as example the commutative algebra got from functions in the line through

$$f(x) \to F \equiv \begin{pmatrix} f(x+\lambda) & 0 \\ 0 & f(x+\epsilon) \end{pmatrix} \tag{3}$$

and put $D = \begin{pmatrix} 0 & \frac{1}{\lambda-\epsilon} \\ \frac{1}{\lambda-\epsilon} & 0 \end{pmatrix}$, then $dx \equiv [D, X] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and we have

$$df \equiv [D, F] = \begin{pmatrix} 0 & \frac{f(x+\epsilon)-f(x+\lambda)}{\lambda-\epsilon} \\ \frac{f(x+\lambda)-f(x+\epsilon)}{\lambda-\epsilon} & 0 \end{pmatrix} = \frac{f(x+\lambda)-f(x+\epsilon)}{\lambda-\epsilon} dx \tag{4}$$

The same result can be formulated by using Majid theory of deformed calculus ([6], with $\epsilon = 0$), and probably also through generalizations of the tangent grupoid [7], but NCG is a more general geometrical theory, and we would prefer to adhere to it.

As for the second piece, a more fundamental interpretation of QFT is needed. The recently noticed connection between perturbative renormalization and Hoft algebras [5] could pave the way for a purely mathematical interpretation of field theory [2]. We could expect to find us against a deformed differential calculus, its Taylor series being Feynman ones, and the infinitesimal limit relaxed to a less exigent one, such as the triviality of the fully renormalized theory.

If we compare the previous example with the actual geometrical representation of the standard model [3] we would suspect that quantization includes the art of keeping the mass ($\frac{1}{\lambda-\epsilon}$) finite while the differences are driven to zero.

5 Wondering if Nature learns calculus.

Lets follow the previous example by adding the generations we suspect to need to discretize curvatures, second derivatives and so on.

To do it, we triple the previous basis, and expand the Dirac operator to hold a whole 3×3 mass matrix M,

$$D = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, M = M^* \tag{5}$$

and repeat the previous procedure.

Lets put an example using an off-diagonal mass matrix,

$$M = \begin{pmatrix} 0 & 1/a & 1/b \\ 1/a & 0 & 1/c \\ 1/b & 1/c & 0 \end{pmatrix} \tag{6}$$

and an initial diagonal matrix

$$F \equiv \begin{pmatrix} f(x+b+c) & & & & & \\ & f(x) & & & & \\ & & f(x+a+b) & & & \\ & & & f(x+a) & & \\ & & & & f(x+a+b+c) & \\ & & & & & f(x+c) \end{pmatrix} \tag{7}$$

Thus,

$$dx \equiv [D, X] = \begin{pmatrix} & & & 0 & 1 & -1 \\ & 0 & & 1 & 0 & 1 \\ & & & -1 & 1 & 0 \\ 0 & -1 & 1 & & & \\ -1 & 0 & -1 & & 0 & \\ 1 & -1 & 0 & & & \end{pmatrix} \tag{8}$$

and (using the notation $[lm...] \equiv f(x + l + m + ...)$ and so on)

$$F' \equiv [D, F]dx^{-1} =$$

$$\frac{1}{2}\begin{pmatrix} \begin{matrix} \frac{[abc]-[bc]}{a} + \frac{[bc]-[c]}{b} & \frac{[abc]-[bc]}{a} - \frac{[bc]-[c]}{b} & \frac{[abc]-[bc]}{a} - \frac{[bc]-[c]}{b} \\ \frac{[a]-[]}{a} - \frac{[c]-[]}{c} & \frac{[a]-[]}{a} + \frac{[c]-[]}{c} & -\frac{[a]-[]}{a} + \frac{[c]-[]}{c} \\ -\frac{[ab]-[a]}{b} + \frac{[abc]-[ab]}{c} & -\frac{[ab]-[a]}{b} + \frac{[abc]-[ab]}{c} & \frac{[ab]-[a]}{b} + \frac{[abc]-[ab]}{c} \end{matrix} & 0 \\ 0 & \begin{matrix} \frac{[a]-[]}{a} + \frac{[ab]-[a]}{b} & \frac{[a]-[]}{a} - \frac{[ab]-[a]}{b} & \frac{[a]-[]}{a} - \frac{[ab]-[a]}{b} \\ \frac{[abc]-[bc]}{a} - \frac{[abc]-[ab]}{c} & \frac{[abc]-[bc]}{a} + \frac{[abc]-[ab]}{c} & -\frac{[abc]-[bc]}{a} + \frac{[abc]-[ab]}{c} \\ -\frac{[bc]-[c]}{b} + \frac{[c]-[]}{c} & -\frac{[bc]-[c]}{b} + \frac{[c]-[]}{c} & \frac{[bc]-[c]}{b} + \frac{[c]-[]}{c} \end{matrix} \end{pmatrix} \tag{9}$$

Just as in the previous example, if we go to the "infinite mass" limit, $a, b, c \to 0$, the matrix F' becomes simply $f' Id_6$, as we would expect. Observe how entries in F' are always composed of samplings of f in three different points, and how the off diagonal terms correspond to numerators of second derivatives, thus going to zero in the limit.

If we continue the process the corresponding F'' matrix (defined as $F'' \equiv [D, F']dx^{-1}$) contains diagonal terms corresponding to second derivatives averaging two samples of two triplets each (for instance,

$${F''}_{4,4} = \frac{1}{4}\left(\frac{\frac{f(x+a+b+c)-f(x+a+b)}{c} - \frac{f(x+a+b)-f(x+a)}{b}}{b} + \frac{\frac{f(x+a)-f(x)}{a} - \frac{f(x+c)-f(x)}{c}}{a}\right),$$

etc.) and off diagonal terms going to zero in the infinite mass limit. Notice that we have still the freedom to impose "mass relations" in the matrix M if we want to control, or even nullify, the off diagonal terms.

Mixing was critical here in order to get a nonzero result, and it seems so in other examples we have essayed. We feel uneasy with this apparent need, as (weak) interactions have not been yet introduced at this level. Anyway, read this toy as an example of the things that NCG can formulate.

6 Conclusions.

The speculative nature of this article does not let us to derive solid conclusions. Instead of this, lets to abstract the sugerences raised along the note:

- The existence of four fundamental particles e, ν, u, d is related to the construction of a coordinate system adequate to Minkowskian space, perhaps Schwarzchild coordinates t, ρ, θ, ϕ.
- Doubling (quirality) of fermions relates to the ambiguity to choose the two points needed to build regularized derivatives.
- Undoubling of neutrino could come from the restriction on the discretization of the radial coordinate.
- Asymptotic freedom (confinement?) is due to the absence of length scale in two of the coordinates, say θ, ϕ
- Tripling of generations comes from the ambiguity to choose the three points needed to build regularized second derivatives.
- Mixing (weak CP etc) of generations could relate also to the restriction on the discretization of second partial derivatives involving ρ (and θ?)
- The Higgs (alternatively space sheets, alternatively antiferromagnetic vacuum) controls the scales which are applied in each derivation.
- Triviality is equivalent to ask for a good limit $\Delta x^i \to 0$ in every derivative.
- Gauge fields will appear if we try to apply the regularization procedure to a general covariant action.

Obviously the risky bet here is the relationship between coordinates and fermions. Once one gets to live with it, each conjecture can try to prove its validity in an independent form. But its combination drives us to look for a strange form of induced gravity: the limit $M_{Higgs} \to \infty$ of the Standard Model should be equivalent to some gravity-like action, and the fermionic matter field would be pumped to existence in its quantum version, as a lateral effect of the renormalization process.

We would like to note that quantum theory is by itself a regularization not of geometry but of variational calculus, as Feynman integral teach us. The nature of the recipe relating deformation and quantization has not been not examined yet (but search [8] for clues), we would expect Plank constant to emerge as the parameter relating both techniques.

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