Connes' tangent groupoid and strict quantization

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Abstract

We address one of the open problems in quantization theory recently listed by Rieffel. By developing in detail Connes' tangent groupoid principle and using previous work by Landsman, we show how to construct a strict flabby quantization, which is moreover an asymptotic morphism and satisfies the reality and traciality constraints, on any oriented Riemannian manifold. That construction generalizes the standard Moyal rule. The paper can be considered as an introduction to quantization theory from Connes' point of view.

1 Introduction

1.1 Motivation

Recently, Rieffel has published a list of open problems [1] in quantization. The main aim of this paper is to address Question 20 of Rieffel's list: "…in what ways can a suitable Riemannian metric on a manifold M be used to obtain a strict deformation quantization of T^*M?" We do that by giving proofs for (a slightly improved variant of) a construction sketched by Connes in Section II.5 of his book [2] on noncommutative geometry, and elsewhere [3]. The paper can be considered as an introduction to the subject of quantization from Connes' point of view and thus serves too a pedagogical purpose.

Rieffel's requirements are stronger that those of formal deformation theory, extant for any Poisson manifold [4]. To our mind, however, the fact that T^*M, for M Riemannian, possesses a strict quantization, was indeed proved by Landsman in the path-breaking paper [5]. Nevertheless, the noncommutative geometry approach presents several advantages, not least that the C^*-theoretical aspects come in naturally. The procedure was suggested by Landsman himself, even before [2] was in print, at the end of his paper. For the sake of simplicity, we deal here with the nonequivariant case only.

The plan of the article is as follows. In this first Section, after introducing groupoids and the tangent groupoid construction, we give an elementary discussion of Connes' recipe for quantization. We find the intersection, for the case $M = \mathbb{R}^n$, of the family of what could be termed "Connes' quantization rules" with the ordinary quantum-mechanical ordering prescriptions. Groupoids are here regarded set-theoretically, questions of smoothness being deferred to Section 2. We show that Moyal's quantization rule belongs to the collection of Connes' quantization rules and in fact is singled out by natural conditions in strict deformation theory.

In Section 2 we spell out our variant of Connes' tangent groupoid construction in full detail. The heart of the matter is the continuity of the groupoid product, for which we give two different proofs.

In Section 3, by means of the mathematical apparatus of Section 2, we restate Landsman's partial answer to Rieffel's question. In particular, we rework the existence proof for a strict quantization *of the Moyal type* (in the sense of being both real and tracial) on Riemannian manifolds. Moreover, using a strong form of the tubular neighbourhood theorem, we show that there exists what Rieffel calls a *flabby* quantization [1]. The paper concludes with a discussion on the C^*-algebraic aspect of the tangent groupoid construction and its relation to the index theorem.

1.2 Basic facts on groupoids

The most economical way to think of a groupoid is as a pair of sets $G^0 \subset G$, and to regard elements of G as arrows and elements of G^0 as nodes.

Definition 1 *A groupoid $G \rightrightarrows G^0$ is a small category in which every morphism has an inverse. Its set of objects is G^0, its set of morphisms is G.*

A group is of course a groupoid with a single object. The gist of the definition is conveyed by the following example.

1.2.0.1 Partial isometries Consider a complex Hilbert space $\mathcal{H}$. The collection of *unitary* arrows between closed subspaces of $\mathcal{H}$ obviously defines a groupoid, for which G is the set of partial isometries $\{ w \in \mathcal{L}(\mathcal{H}) : ww^*w = w \}$ and G^0 is the set of orthogonal projectors in $\mathcal{L}(\mathcal{H})$. Recall that, given w, we can write

$$\mathcal{H} = \ker w \oplus \operatorname{im} w^* = \ker w^* \oplus \operatorname{im} w.$$

Hence w^*w is the orthogonal projector with range $\operatorname{im} w^*$, while ww^* is the orthogonal projector with range $\operatorname{im} w$; of course, w^* is the inverse of w. We naturally identify two maps r, s (respectively "range" and "source") from G to G^0: $r(w) := ww^*$ and $s(w) := w^*w$. Also, it is natural to consider

$$G^{(2)} := \{ (u, v) \in G \times G : u^*u = vv^* \}.$$

This defining equation is a sufficient condition for the operator product uv to be a partial isometry, as follows from the simple calculation

$$uv(uv)^*uv = uvv^*u^*uv = uvv^*v = uv.$$

The example motivates a more cumbersome restatement, that includes all the practical elements of the definition.

Definition 2 *A groupoid $G \rightrightarrows G^0$ consists of: a set G, a set G^0 of "units" with an inclusion $G^0 \hookrightarrow G$, two maps $r, s : G \to G^0$, and a composition law $G^{(2)} \to G$ with domain*

$$G^{(2)} := \{ (g, h) : s(g) = r(h) \} \subseteq G \times G,$$

subject to the following rules:

(1) if $g \in G^0$ then $r(g) = s(g) = g$;
(2) $r(g)g = g = gs(g)$;
(3) each $g \in G$ has an "inverse" g^{-1}, satisfying $gg^{-1} = r(g)$ and $g^{-1}g = s(g)$;
(4) $r(gh) = r(g)$ and $s(gh) = s(h)$ if $(g, h) \in G^{(2)}$;
(5) $(gh)k = g(hk)$ if $(g, h) \in G^{(2)}$ and $(gh, k) \in G^{(2)}$.

The examples of groupoids we shall use have a differential geometric flavour instead.

1.2.0.2 A vector bundle $E \xrightarrow{\pi} M$ Here $G = E$ is the total space, $G^0 = M$ is the base space, $r = s = \pi$ so that $G^{(2)} = \biguplus_{x \in M} E_x \times E_x$ (the total space of the Whitney sum $E \oplus E$), and the composition law is *fibrewise addition*.

1.2.0.3 The double groupoid of a set Given a set M, take $G = M \times M$ and $G^0 = M$, included in $M \times M$ as the diagonal subset $\Delta(M) := \{ (x,x) : x \in M \}$. Define $r(x,y) := x$, $s(x,y) := y$. Then $(x,y)^{-1} = (y,x)$ and the composition law is

$$(x,y) \cdot (y,z) = (x,z).$$

We shall generally call G^0 the *diagonal* of G.

1.3 Connes' tangent groupoids

We recall some basic concepts of differential geometry, particularly sprays and normal bundles, that we shall later use.

Given a symmetric linear connection on a differentiable manifold M, one can define a vector field Γ on TM whose value at $v \in TM$ is its horizontal lift to $T_v(TM)$. This vector field is called the geodesic spray of the connection and its integral curves are just the natural lift of geodesics in M. The geodesic spray is a second-order differential equation vector field satisfying an additional condition of degree-one homogeneity which corresponds to the affine reparametrization property of geodesics. More generally, a second-order differential equation vector field is said to be a spray if the set of its integral curves is invariant under any affine reparametrization: these curves are called geodesics of the spray. Given a spray Γ, there is a symmetric connection whose geodesic spray is Γ; and conversely, the connection is fully determined by its geodesic spray, that can also be used to construct the exponential map $\exp \colon T_xM \to M$. A Riemannian structure on M determines one symmetric linear connection, the Levi-Civita connection, and consequently a Riemannian spray.

Now let Y^0 be a submanifold of a manifold Y. The normal bundle $\mathcal{N}^Y_{Y^0}$ to Y^0 in Y is defined as the vector bundle $\mathcal{N}^Y_{Y^0} := TY/TY^0$, where the notation means that its base is Y^0 and its fibre is given by the equivalence classes of the elements of the tangent bundle TY under the relation: $X_1 \sim X_2$, for $X_1, X_2 \in T_qY$ with $q \in Y^0$, if and only if $X_1 = X_2 + V$ for some $V \in T_qY^0$. The usual way to work with such a structure is to choose a representative in each class, thereby forming a complementary bundle to TY^0 in TY restricted to Y^0. There is no canonical choice for the latter, in general. When a Riemannian metric is provided on Y, there is a natural definition of the complementary bundle as the orthogonal complement of the tangent space T_qY^0 in T_qY (although other choices may be convenient, even in the Riemannian case). Once we have chosen a suitable representative of each class, the bundle $\mathcal{N}^Y_{Y^0}$ becomes a subbundle of TY and we can consider the exponential map exp of TY restricted to $\mathcal{N}^Y_{Y^0}$.

We recall also that a *tubular neighbourhood* of Y^0 in Y is a vector bundle $E \to Y^0$, an open neighbourhood Z of its zero section and a diffeomorphism of Z onto an open set (the tube) $U \subset Y$ containing Y^0, which restricts over the zero section to the inclusion of Y^0 in Y [6]. We say that the tubular neighbourhood is *total* when $Z = E$. The main theorem in this context establishes that, given a spray on Y, one can always construct a tubular neighbourhood by making use of the corresponding exponential map. When the spray is associated to a Riemannian metric, one can always have a total tubular neighbourhood, because a Euclidean bundle is compressible, i.e., isomorphic as a fibre bundle to an open neighbourhood of the zero section: see [6].

For suitable $\hbar_0 > 0$, one can then define the normal cone deformation (a sort of blowup in the differentiable category) of the pair (Y, Y^0), denoted $\mathcal{M}^Y_{Y^0}$, by gluing together $Y \times (0, \hbar_0]$ with $\mathcal{N}^Y_{Y^0}$ as a boundary with the help of the tubular neighbourhood construction [7]. The construction is particularly interesting when (Y, Y^0) is a groupoid, in that it gives a "normal groupoid" with diagonal $Y^0 \times [0, \hbar_0]$.

We consider a particular case of this construction, the tangent groupoid. Let M now be an orientable Riemannian manifold. Denote by $\mathcal{N}^\Delta$, rather than $\mathcal{N}^{M\times M}_M$, the normal bundle associated to the diagonal embedding $\Delta\colon M \to M \times M$. We can identify $TM \oplus TM$ with the restriction of $T(M \times M)$ to $\Delta(M)$, and the tangent bundle over M is identified to

$$\{\, (\Delta(q), X_q, X_q) : (q, X_q) \in TM \,\};$$

thereby the normal bundle $\mathcal{N}^\Delta$ to M in $M \times M$ can *a priori* be identified with

$$\{\, (\Delta(q), \varphi_{1q} X_q, \varphi_{2q} X_q) : (q, X_q) \in TM \,\}, \tag{1}$$

where $\varphi_1, \varphi_2 \in \mathrm{End}(TM)$ are any two bundle endomorphisms (i.e., continuous vector bundle maps from TM into itself) such that the linear map $\varphi_{1q} - \varphi_{2q}$ on T_qM is invertible for all $q \in M$; this we write as $\varphi_1 - \varphi_2 \in GL(TM)$. We shall assume, for definiteness, that each $\varphi_{1q} - \varphi_{2q}$ is homotopic to the identity: in other words, that there is an isomorphism of oriented vector bundles between TM and $\mathcal{N}^\Delta$.

The tangent groupoid $G_M \rightrightarrows G^0_M$, according to Connes, is essentially the normal groupoid $\mathcal{M}^{M\times M}_M$ modulo that isomorphism [2]. That is to say, we think of the disjoint union $G_M = G_1 \uplus G_2$ of two groupoids

$$G_1 := M \times M \times (0, \hbar_0], \qquad G_2 := TM,$$

where G_1 is the disjoint union of copies $M \times M \times \{\hbar\}$ of the double groupoid of M, parametrized by $0 < \hbar \leq \hbar_0$. The compositions are

$$
\begin{aligned}
(x, y, \hbar) \cdot (y, z, \hbar) &= (x, z, \hbar) \qquad && \text{with } \ 0 < \hbar \leq \hbar_0, \\
(q, X_1) \cdot (q, X_2) &= (q, X_1 + X_2) \qquad && \text{if } \ X_1, X_2 \in T_q M.
\end{aligned}
$$

The topology on G_M is such that, if $(x_n, y_n, \hbar_n)$ is a sequence of elements of G_1 with $\hbar_n \downarrow 0$, then it converges to a tangent vector (q, X) iff

$$x_n \to q, \quad y_n \to q$$

and

$$\frac{x_n - y_n}{\hbar_n} \to X. \tag{2}$$

The last condition, although formulated in a local chart, clearly makes intrinsic sense, as $x_n - y_n$ and X get multiplied by the same factor, to first order, under a change of charts. It forces us to demand that $\varphi_{1q} - \varphi_{2q}$ be equal to the identity in (1); this will be seen later to be necessary also for linear symplectic invariance of the associated quantization recipe in the flat case. Actually Connes picks, to coordinatize the tangent groupoid, the particular identification given by $\varphi_{1q} X_q = 0$, $\varphi_{2q} X_q = -X_q$.

1.4 Connes' quantization rule on $\mathbb{R}^n$

Consider then the case $M = \mathbb{R}^n$. The groupoid $G_{\mathbb{R}^n}$ is certainly a manifold with boundary $T\mathbb{R}^n$. The neighbourhood of points in the boundary piece is described as follows: for $(q, X) \in T\mathbb{R}^n$, define

$$(x, y, \hbar) := \Phi_\hbar(q, X, \hbar) = (q + \hbar\varphi_1 X, q + \hbar\varphi_2 X, \hbar),$$

with $\varphi_1 X - \varphi_2 X = X$. (It is natural in this context to take φ_{1q}, φ_{2q} to be constant maps.) In fact, Φ is a linear isomorphism of $T\mathbb{R}^n \times [0, 1]$ onto the groupoid.

Now comes Connes' quantization recipe: a function on G_M is first of all a pair of functions on G_1 and G_2 respectively. The first one is essentially the kernel for an operator on $L^2(M)$, the second the Fourier transform of a function on the cotangent bundle T^*M, i.e., the classical phase space. The condition that both match seamlessly to give a regular function on G_M is precisely construed as the quantization rule!

Let $a(q,p)$ be a function on $T^*\mathbb{R}^n$. Its inverse Fourier transform in the second variable gives us a function on $T\mathbb{R}^n$:

$$\mathcal{F}^{-1}a(q,X) = \int_{\mathbb{R}^n} e^{\mathrm{i}Xp} a(q,p)\, \mathrm{d}p.$$

To the function a, Connes' prescription associates then the following family of kernels:

$$k_a(x,y;\hbar) := \mathcal{F}^{-1}a(\Phi_\hbar^{-1}(x,y,\hbar)).$$

The map $a \mapsto k_a$ is thus linear. We get the dequantization rule by Fourier inversion:

$$a(q,p) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} k_a(q+\hbar\varphi_1 X, q+\hbar\varphi_2 X;\hbar)\, e^{-\mathrm{i}pX}\, \mathrm{d}X. \tag{3}$$

We may well ask where Connes' rule stands with respect to the usual quantization rules. We find it more convenient to argue from the dequantization formula. Any linear dequantization rule can be expressed in the form

$$a(q,p) = \int_{\mathbb{R}^{2n}} K(q,p,x,y;\hbar)\, k_a(x,y;\hbar)\, \mathrm{d}x\, \mathrm{d}y,$$

for a suitable distributional kernel K. To select useful rules, one seeks to impose reasonable conditions. The simplest is obviously equivariance under translations in $T^*\mathbb{R}^n$, corresponding physically to Galilei invariance. Equivariance under translation of the spatial coordinates amounts immediately to the condition

$$\Big(\frac{\partial}{\partial q} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big)K = 0, \tag{4}$$

i.e., K depends only on the combinations $x-y$ and $q-\frac{1}{2}(x+y)$. A similar argument in the dual space shows that equivariance under translation of the momenta amounts to

$$K(q,p+p',x,y;\hbar) = e^{-\mathrm{i}p'(x-y)/\hbar} K(q,p,x,y;\hbar). \tag{5}$$

As beautifully discussed in [8], both conditions together call for the role of Weyl operators in quantization. In fact, it is clear that the most general kernel satisfying (4) and (5) is of the form

$$K(q,p,x,y;\hbar) = \frac{1}{(2\pi\hbar)^{2n}} e^{-\mathrm{i}p(x-y)/\hbar} \int_{\mathbb{R}^n} f(\theta, x-y) \exp(-\mathrm{i}\theta(q - \tfrac{1}{2}(x+y))/\hbar)\ \mathrm{d}\theta,$$

where f should have no zeros, on account of invertibility; that can be rewritten as

$$K(q,p,x,y;\hbar) = \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} f(\theta,\tau) e^{-\mathrm{i}(\theta q + \tau p)/\hbar} \langle y \mid \exp(-\mathrm{i}(\theta Q + \tau P)/\hbar) x\rangle\ \mathrm{d}\theta\ \mathrm{d}\tau,$$

where Q, P denote the usual quantum observables for position and momentum, respectively, and $\exp(-\mathrm{i}(\theta Q + \tau P)/\hbar)$ are the Weyl operators.

Further requirements on the quantization rule lead to a sharpening in the determination of f. For instance, in the terminology of [10], the requirement of *semitraciality* (that dequantization of the kernel corresponding to the identity operator be the function 1) leads to $f(0,0) = 1$; of *reality* (that the kernels for real classical observables correspond to selfadjoint operators (quantum observables according to Quantum Mechanics postulates) forces $f(\theta,\tau) = f^*(-\theta,-\tau)$, and so on. Moyal's rule [11], that is, the paradigmatic example of strict deformation quantization, corresponds to taking $f = 1$.

A very important requirement, still in the terminology of [10], is *traciality*; that is, the coincidence between the classical and quantum averages for the product of observables. Mathematically it is equivalent to unitarity of K, which demands $|f|^2 = 1$. Taken together, traciality and reality allow to rewrite quantum mechanics as a statistical theory in classical phase space; and mathematically, they practically force the Moyal rule. Traciality is further discussed in Section 3, in the general context of Riemannian manifolds.

On the other hand, Connes' dequantization rule (3) gives

$$K(q,p,x,y;\hbar) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \delta(x - q - \hbar\varphi_1 X)\delta(y - x + \hbar X) \exp(-\mathrm{i}pX)\ \mathrm{d}X.$$

This integral can be rewritten as

$$(2\pi\hbar)^{-n}\, \delta(x - q - \varphi_1(x-y))\, e^{-\mathrm{i}p(x-y)/\hbar},$$

and further transformed into

$$\frac{1}{(2\pi\hbar)^{2n}} e^{-\mathrm{i}p(x-y)/\hbar} \int_{\mathbb{R}^n} \exp(\mathrm{i}\theta(\tfrac{1}{2}(x+y) - q + (\tfrac{1}{2} - \varphi_1)(x-y))/\hbar)\ \mathrm{d}\theta,$$

which is of the previously given form, with

$$f(\theta, \tau) = \exp(\mathrm{i}\theta(\tfrac{1}{2} - \varphi_1)\tau/\hbar).$$

We conclude that Connes' quantization rule, in the flat space case, is determined up to an arbitrary linear transformation of $\mathbb{R}^n$.

Had we tried to keep the most general formula

$$K(q,p,x,y;\hbar) = \frac{1}{(2\pi\hbar)^n} \int\limits_{\mathbb{R}^n} \delta(x-q-\hbar\varphi_{1q}X)\delta(y-x+\hbar(\varphi_{1q}-\varphi_{2q})X) \exp(-\mathrm{i}pX)\, \mathrm{d}X,$$

we would have had two difficulties: equivariance under translation of the position would be lost since $\varphi_{1q} \neq \varphi_{1q'}$ in general, and equivariance under translation of the momentum would be lost since $\varphi_{1q} - \varphi_{2q} \neq \mathrm{id}_{T_q\mathbb{R}^n}$ in general; this bears on the physical meaning of Connes' limit condition (2).

It would be tempting, also in view of (2), to add the time variable to the mathematical apparatus of this paper, in the spirit of Feynman's formalism, and to study the interchangeability of the limits $t \downarrow 0$ and $\hbar \downarrow 0$.

Happily, Moyal's rule is included among Connes' rules: it follows from the most natural choice $\varphi_1 = \frac{1}{2}$, $\varphi_2 = -\frac{1}{2}$ (see the discussion in [12]). We arrive at the following conclusion.

Proposition 3 *Moyal quantization rule is the only* ***real*** *quantization of the Connes type, in the case $M = \mathbb{R}^n$.*

Connes' own choice in [2] is $\varphi_1 = 0$, $\varphi_2 = -1$; this corresponds to the "standard" ordering prescription, in which the quantization of $q^n p^m$ is $Q^n P^m$; whereas the choice $\varphi_1 = 1$, $\varphi_2 = 0$ leads to the "antistandard" ordering, in which the quantization of $p^m q^n$ is $P^m Q^n$. Note that all of Connes' prescriptions are tracial. In order to obtain the "normal" and "antinormal" prescriptions of use in field theory, which are real but not tracial, one would have to complexify Connes' construction; we shall not go into that.

Following Landsman, we will strengthen the definition of strict quantization in order to remain in the real context (see Section 3). Our elementary discussion in this subsection leads to presume that real strict quantization can be done in the framework of tangent groupoids; this is tantamount to the generalization of the Moyal rule to arbitrary manifolds endowed with sprays. All the pertinent differential geometric constructions for that purpose are taken up in the next section.

2 The tangent groupoid construction: the Moyal version

Definition 4 *A* ***smooth groupoid*** *is a groupoid* (G, G^0) *together with differentiable structures on* G *and* G^0 *such that the maps* r *and* s *are submersions, and the inclusion map* $G^0 \hookrightarrow G$ *is smooth as well as the product* $G^{(2)} \to G$.

Note the dimension count: if $\dim G = n$, $\dim G^0 = m$, then $\dim G^{(2)} = 2n - m$.

The definition of smooth groupoid is taken from [2]. Now we establish his Proposition II.5.4, there left unproven by Connes.

Proposition 5 *The tangent groupoid* G_M *to a smooth manifold* M *is a smooth groupoid.*

2.1 G_M *as a manifold with boundary*

The groupoid $G_1 = M \times M \times (0, \hbar_0]$, that will be the interior of G_M (plus a trivial "outer" boundary, which we neglect to mention in our following arguments) is given the usual product manifold structure. To complete the definition of the manifold structure of G, consider the isomorphism from TM to N^Δ given by $(q, X_q) \mapsto (q, \frac{1}{2}X_q, -\frac{1}{2}X_q)$. Consider also the product manifold $TM \times [0, \hbar_0]$. We choose a spray on M —this is provided, for instance, by a choice of Riemannian metric on M— and define a map

$$TM \times [0, \hbar_0] \supset U \xrightarrow{\Phi} G_M,$$

where U is open in $TM \times [0, \hbar_0]$ and includes $TM \times \{0\}$, as follows:

$$\Phi(q, X_q, \hbar) := (\exp_q(\hbar X_q/2), \exp_q(-\hbar X_q/2), \hbar) \qquad \text{for} \quad \hbar > 0,$$

and

$$\Phi(q, X_q, 0) := (q, X_q) \qquad \text{for} \quad \hbar = 0.$$

Here exp denotes the exponential map associated to the spray: we know that, for a fixed $\hbar$, the exponential map defines a diffeomorphism of an open neighbourhood V of the zero section of TM onto an open neighbourhood of the diagonal in $M \times M$; and we decide that a point $(q, X_q, \hbar)$ is in U if $\Phi(q, X_q, \hbar)$ is contained in V.

Therefore, both the existence and, for a suitable choice of U, the bijectivity of the map Φ follow from the tubular neighbourhood theorem. As U is an open (sub)manifold with boundary, we can carry the structure of manifold with boundary to G_M, obtaining that TM is the boundary of the groupoid G_M in the topology associated to that structure. The diagonal is obviously $M \times [0, \hbar_0]$.

We remind the reader that Connes uses instead the chart given by

$$(q, X_q, \hbar) \mapsto (q, \exp_q(-\hbar X_q), \hbar) \qquad \text{for} \quad \hbar > 0,$$

and

$$(q, X_q, 0) \mapsto (q, X_q) \qquad \text{for} \quad \hbar = 0.$$

From now on, a particular Riemannian structure is assumed chosen, and when convenient we shall also assume, as we then may, that U is the whole of $TM \times [0, \hbar_0]$. To continue the proof of smoothness of G_M, we need to check that the various mappings have the required smoothness properties. The basic idea is to pull all maps on G_M back to $TM \times [0, \hbar_0]$ and prove smoothness there.

Consider first the inclusion map $i\colon G_M^0 \to G_M$. It is obvious that the restriction of i to $M \times (0, \hbar_0]$ is smooth in its domain. If we now consider its restriction to $i^{-1}(\Phi(U))$ and compose it with Φ^{-1}, we obtain a map which can be written as:

$$\Phi^{-1} \circ i(x, \hbar) = \Phi^{-1}(x, x, \hbar) = (x, 0_x, \hbar) \qquad \text{for} \quad \hbar > 0,$$

and

$$\Phi^{-1} \circ i(x, 0) = \Phi^{-1}(x, 0_x, 0) = (x, 0_x, 0) \qquad \text{for} \quad \hbar = 0.$$

This map is obviously smooth in its domain and, as Φ is a diffeomorphism, i is smooth in its domain.

We shall consider now the range and source maps. The smoothness of both maps when restricted to $M \times M \times (0, \hbar]$ is again obvious, as is the fact that they are of maximum rank (hence submersions when restricted to this domain). The composition of Φ with the restriction of r to $\Phi(U)$ is expressed as:

$$r \circ \Phi(q, X_q, \hbar) = r(\exp_q(\hbar X_q/2), \exp_p(-\hbar X_p/2), \hbar) = (\exp_q(\hbar X_q/2), \hbar)$$

for $\hbar > 0$ and

$$r \circ \Phi(q, X_q, 0) = (q, 0)$$

for $\hbar = 0$. Again this map is smooth and of maximum rank in its domain, so that r must also be a submersion. The corresponding proof for the source map is analogous.

2.2 The geometrical structure of $G_M^{(2)} \subset G_M \times G_M$

To define a differentiable structure for $G_M^{(2)}$, we proceed as in the previous case, by defining a bijection between an open set in $G_M^{(2)}$ and an open set in a manifold with boundary, and transporting the differential structure via the bijection.

What kind of manifold is $G_M^{(2)}$? The product has to be a smooth mapping between two manifolds with boundary, mapping the boundary on the boundary and the interior on the interior. Thus, from the definition of the product it is clear that the boundary of the manifold should be the Whitney sum $TM \oplus TM$, arising as the pullback with respect to the diagonal injection of M in $M \times M$ of the product bundle $TM \times TM$.

Points in the interior of the manifold are pairs of the form: $((x, y, \hbar), (y, z, \hbar))$. Therefore, we need to define a differential structure in $G_M^{(2)}$ in such a way that it becomes a manifold with boundary $TM \oplus TM$, its interior being diffeomorphic to $M \times M \times M \times (0, \hbar_0]$.

Consider now $TM \times TM \times [0, \hbar_0]$. This is a manifold with boundary, with a natural differential structure. Let $(q', q, X_{q'}, Y_q, \hbar)$ be a point in this manifold; we use our construction for G_M on each TM separately, i.e., we set the bijection:

$$(q', q, X_{q'}, Y_q, \hbar) \leftrightarrow \{(e_{q'}^{\hbar X_{q'}/2}, e_{q'}^{-\hbar X_{q'}/2}, \hbar), (e_q^{\hbar Y_q/2}, e_q^{-\hbar Y_q/2}, \hbar)\}$$

for $\hbar > 0$ —with an obvious notation for exp. Those points that interest us result just from imposing that $e_{q'}^{-\hbar X_{q'}} = e_q^{\hbar Y_q}$. We shall see that this constraint defines a regular submanifold of $TM \times TM \times [0, \hbar_0]$. For that, for a fixed value of $\hbar$, consider the sequence of maps

$$\Psi : TM \times TM \xrightarrow{\Phi} M \times M \times M \times M \xrightarrow{\pi_{23}} M \times M \xrightarrow{\Phi^{-1}} TM \to \mathbb{R}^n$$

given by

$$\begin{aligned}(q', q, X_{q'}, Y_q) &\mapsto (e_{q'}^{\hbar X_{q'}/2}, e_{q'}^{-\hbar X_{q'}/2}, e_q^{\hbar Y_q/2}, e_q^{-\hbar Y_q/2}) \\ &\mapsto (e_{q'}^{-\hbar X_{q'}/2}, e_q^{\hbar Y_q/2}) \mapsto (r, X_r) \mapsto X_r,\end{aligned}$$

where the (r, X_r), that depend on $\hbar$, are found so $\Phi(r, X_r) = (e_{q'}^{-\hbar X_{q'}/2}, e_q^{\hbar Y_q/2})$. This composition defines a differentiable mapping of constant rank (it is composed of two bijections and two projections onto factors of a product). We then extend Ψ to a map from $TM \times TM \times [0, \hbar_0]$ to $\mathbb{R}^n \times [0, \hbar_0]$ and have then that $\Psi^{-1}(\{0\} \times [0, \hbar_0])$ defines a regular submanifold S of $TM \times TM \times [0, \hbar_0]$, whose differentiable structure we use to define the structure of manifold with boundary on $G^{(2)}$. [It is perhaps not entirely clear that the boundary of S is $TM \oplus TM$, as we wish. But remember that r depends on $\hbar$ through the above manipulations. If we have a sequence

$$(q'_n, q_n, X_{q'_n}, Y_{q_n}, \hbar_n)$$

in S, with $\hbar_n \downarrow 0$, its limit is a point (s, s, X, Y) in $TM \times TM$ where $s = \lim q'_n = \lim q_n = \lim r(\hbar_n)$.]

2.3 Continuity of the product

A very simple argument with Riemannian flavour allows one to prove at least continuity of the product operation in the tangent groupoid. Consider a sequence in $G^{(2)}$:

$$\{(x_n, y_n, \hbar_n), (y_n, z_n, \hbar_n)\} = \Phi(q'_n, q_n, X_n, Y_n, \hbar_n)$$

with limit on the boundary. We need to check that the limit of the products coincides with the product of the limits in G_M. For elements close enough to the boundary, we know that $(s(n), Z_s(n))$ exists, so

$$x_n = e_{s(n)}^{\hbar Z_s(n)/2}, \quad z_n = e_{s(n)}^{-\hbar Z_s(n)/2}.$$

Assume that

$$q'_n \to s, \quad q_n \to s, \quad X_n \to A, \quad Y_n \to B \qquad \text{as } \hbar \downarrow 0.$$

Now, we have

$$(e_{s(n)}^{\hbar Z_s(n)/2}, e_{s(n)}^{-\hbar Z_s(n)/2}) = (e_{q'_n}^{\hbar X_n/2}, e_{q_n}^{-\hbar Y_n/2}) \to (s, s) \qquad \text{as } \hbar \downarrow 0,$$

hence

$$\lim s(n) = s.$$

It remains to show that $\lim Z_s(n) = A + B$. This will follow if we prove $Z_n = X_n + Y_n + o(\hbar_n)$. But that follows from consideration of the small triangle with vertices x_n, y_n, z_n, formed by geodesics through q'_n, q_n, $s(n)$ with directions $\pm X_n$, $\pm Y_n$, $\pm Z_s(n)$.

One sees that $2(X_n + Y_n - Z_s(n))$ is approximately a circuit around this triangle; by the Gauss–Bonnet theorem, we conclude that

$$X_n + Y_n - Z_s(n) \approx O(\hbar^2).$$

2.4 A functorial proof of smoothness

Lemma 6 *Let there be given two closed submanifolds $X_0 \hookrightarrow X$ and $Y_0 \hookrightarrow Y$ and a smooth mapping $f \colon X \to Y$ which satisfies $f(X_0) \subset Y_0$. Then the induced mapping $\tilde{f} \colon \mathcal{M}^X_{X^0} \to \mathcal{M}^Y_{Y^0}$ between the corresponding normal cone deformations is also smooth.*

This is Lemma 2.1 in [13], where no proof is offered. We give some details of the lemma and then of its application. First of all, if f is a smooth mapping from X to Y such that $f(X_0) \subset Y_0$, the image under f_* of the tangent bundle to X_0 is a subbundle of TY_0, implying the existence of an induced mapping between the respective normal bundles, that we shall continue to call f_*. Now we define $\tilde{f}$ by

$$\tilde{f}(x, \hbar) := (f(x), \hbar) \qquad \text{for} \quad \hbar > 0,$$

and

$$\tilde{f}(a, X_a) := (f(a), f_*(X_a))$$

for (a, X_a) an element of the normal bundle and $\hbar = 0$. Now, differentiability of $\tilde{f}$ follows from the limit

$$\lim_{\hbar \downarrow 0} \hbar^{-1} \exp^{-1}_{f(a)}[f(\exp_a(\hbar X_a))] = f_*(X_a).$$

The application to the groupoid operations in our context is plain. We discuss the product operation, the only relatively tricky one. Let $m \colon G^{(2)} \to G$ be the groupoid multiplication. Now $G^{(2)} \cap (G^0 \times G^0) = \Delta(G^0)$ and $m(G^{(2)} \cap (G^0 \times G^0)) = G^0$; indeed, if $(u, v) \in G^{(2)} \cap (G^0 \times G^0)$ then $u = s(u) = r(v) = v$ by property (i) of Definition 2, and so $uv = us(u) = u$ by property (ii). Therefore $\tilde{m} \colon \mathcal{M}^{G^{(2)}}_{G^0} \to \mathcal{M}^G_{G^0}$ is smooth. It remains to prove that $\mathcal{M}^{G^{(2)}}_{G^0}$ is diffeomorphic to $(\mathcal{M}^G_{G^0})^{(2)}$. But this is clear on examining the definitions: indeed,

$$\mathcal{M}_{G^0}^{G^{(2)}} = G^{(2)} \times (0, \hbar_0] \uplus \mathcal{N}_{G^0}^{G^{(2)}}$$
$$= \{ (g, h, \hbar) : s(g) = r(h) \} \uplus \{ (u, X_u, Y_u) : X_u, Y_u \in \mathcal{N}_{G^0}^{G} \},$$

whereas

$$(\mathcal{M}_{G^0}^{G})^{(2)} = \{ (g, \hbar_1; h, \hbar_2) : s(g) = r(h),\ \hbar_1 = \hbar_2 \} \uplus \{ (u, X_u; v, Y_v) : u = v \}.$$

For $G = G_M$, that boils down to $TM \oplus TM \approx \mathcal{N}_M^{M\times M\times M}$. Note finally the dimension count: the dimension of $(\mathcal{M}_{G^0}^{G})^{(2)}$ is $2(\dim G + 1) - \dim(G^0 + 1) = 2\dim G - \dim G^0 + 1$, which is clearly the same as the dimension of $\mathcal{M}_{G^0}^{G^{(2)}}$.

This completes the proof of Proposition 5.

3 Tangent groupoids and strict quantization

3.1 The deformation conditions

In our definition of quantization, we actually strengthen some of Rieffel's requirements. Regard T^*M as a Poisson manifold and consider the classical C^*-algebra $A_0 := C_0(T^*M)$ of continuous functions vanishing at infinity. We choose a dense subalgebra $\mathcal{A}_0$ (there is considerable freedom in that, but, to fix ideas, we think of the functions whose Fourier transform in the second argument has compact support), and we search for a family of mappings $Q_\hbar$ into noncommutative C^*-algebras $A_\hbar$ such that the following relations hold for arbitrary functions in $\mathcal{A}_0$:

(1) the map $\hbar \mapsto \|Q_\hbar(f)\|$ is continuous on $[0, h_0)$ with $Q_0 = I$;
(2) $\lim_{\hbar\to 0} \|Q_\hbar(f_1)Q_\hbar(f_2) - Q_\hbar(f_2)Q_\hbar(f_1) - i\hbar Q_\hbar(\{f_1, f_2\})\| = 0$;
Those are Rieffel's *strict quantization* conditions (Question 23 of [1]), to which we add:
(3) the *asymptotic morphism* condition $\lim_{\hbar\to 0} \|Q_\hbar(f_1)Q_\hbar(f_2) - Q_\hbar(f_1 f_2)\| = 0$;
(4) the *reality* condition $Q_\hbar(f^*) = Q_\hbar(f)^*$; and also
(5) the *traciality* condition $\mathrm{Tr}[Q_\hbar(f_1)Q_\hbar(f_2)] = \int_{T^*M} f_1(q,p) f_2(q,p)\, d\mu_\hbar(q,p)$;

where we use the same symbols $*$ (a bit overworked, admittedly) and $\|\cdot\|$ for the adjoint and norm in every C^*-algebra.

Axioms (1) to (4) are a slight variant of Landsman's axioms. That the tangent groupoid construction provides an answer to the twentieth query by Rieffel follows indeed from Landsman's calculations in [5]. There is no point in repeating them here, and we limit ourselves to the necessary remarks to fit them in the tangent groupoid framework. Axiom (5) is employed to further select a unique recipe.

In [1], Rieffel introduces the important concept of "flabbiness": a deformation quantization is *flabby* if it contains the algebra of smooth functions of compact support on M. The constructions performed in this paper require in principle only the existence of sprays, in order to use the tubular neighbourhood theorem. However, in that case flabbiness is not guaranteed (see below).

3.2 The C^*-algebra of a groupoid

The natural operation on functions of a groupoid is *convolution*:

$$(a * b)(g) := \int_{\{hk=g\}} a(h)\, b(k) = \int_{\{h:r(h)=r(g)\}} a(h)\, b(h^{-1}g)$$

but for this to make sense we need a measure to integrate with. We can either define a family of measures on the fibres of the map r, $G^x := \{\, g \in G : r(g) = x \,\}$ for $x \in G^0$ (see the detailed treatments given by Kastler [14] and Renault [15]) or we can finesse the issue by ensuring that the integrand is always a 1-density on each G^x.

In the second approach, one uses half-densities, rather than functions. We summarize it here, for completeness. Denote the typical fibre of s by $G_y := \{\, g \in G : s(g) = y \,\}$ for $y \in G^0$. Since r and s are submersions, the fibres G^x and G_y are submanifolds of G of the same dimension, say k. If $x = r(g)$ and $y = s(g)$, then $\Lambda^k T_g G^x$ and $\Lambda^k T_g G_y$ are lines. Let $\Omega_g^{1/2}$ be the set of maps

$$\rho : \Lambda^k T_g G^x \otimes \Lambda^k T_g G_y \to \mathbb{C} \qquad \text{such that} \quad \rho(t\alpha) = |t|^{1/2} \rho(\alpha) \quad \text{for } t \in \mathbb{R}.$$

This is a (complex) line, and it forms the fibre at g of a line bundle $\Omega^{1/2} \to G$, called the "half-density bundle". Let $C_c^\infty(G, \Omega^{1/2})$ be the space of smooth, compactly supported sections of this bundle. For a, b in this space, the convolution formula makes sense and $a * b \in C_c^\infty(G, \Omega^{1/2})$ also. The C^*-algebra of the smooth groupoid $G \rightrightarrows G^0$ is the algebra $C^*(G)$ obtained by completing $C_c^\infty(G, \Omega^{1/2})$ in the norm $\|a\| := \sup_{y \in G^0} \|\pi_y(a)\|$, where π_y is the representation of $C_c^\infty(G, \Omega^{1/2})$ on the Hilbert space $L^2(G_y, \Omega^{1/2})$ of half-densities on the s-fibre G_y:

$$\pi_y(a)\xi : g \mapsto \int_{G_y} a(h)\, \xi(h^{-1}g)$$

where one notices that the integrand is a 1-density on G_y. If $G = M \times M$, we get just the convolution of kernels:

$$(a * b)(x, z) := \int_M a(x, y)\, b(y, z)\, \mathrm{d}y$$

where $\mathrm{d}y$ denotes integration of a 1-density on M parametrized by y.

This business of half-densities is very canonical and independent of preassigned measures. However, in our case, if M is an oriented Riemannian manifold, we may use the volume form $\mathrm{d}\nu(x) := \sqrt{g(x)}\, \mathrm{d}x$, where $g(x) := \det[g_{ij}(x)]$, to replace half-densities by square-integrable functions on $M \times M$ and on M; thus $C^*(M \times M)$ is the completion of $C_c^\infty(M \times M)$ acting as integral kernels on $L^2(M)$, so that $C^*(M \times M) \simeq \mathcal{K}$, the C^*-algebra of compact operators. If $G = TM$ is the tangent bundle, then $C^*(TM)$ is the completion of the convolution algebra

$$(f_1 * f_2)(q, X) := \int_{T_qM} f_1(q, Y)\, f_2(q, X - Y)\, \sqrt{g(q)}\, \mathrm{d}Y$$

where we may take $f_1(x, \cdot)$ and $f_2(x, \cdot)$ in $C_c^\infty(T_xM)$. The *Fourier transform*

$$\mathcal{F}a(q, p) = \frac{1}{(2\pi)^n} \int_{T_qM} e^{-\mathrm{i}pX} a(q, X)\, \sqrt{g(q)}\, \mathrm{d}X$$

replaces convolution by the ordinary product on the total space T^*M of the cotangent bundle. This extends to the isomorphism also called $\mathcal{F}\colon C^*(TM) \to C_0(T^*M)$, with inverse:

$$\mathcal{F}^{-1}b(q, X) = \int_{T_q^*M} e^{\mathrm{i}pX} b(q, p) \frac{\mathrm{d}^n p}{\sqrt{g(q)}}.$$

3.3 The quantization and dequantization recipes

Let $\gamma_{q,X}$ be the geodesic on M starting at q with velocity X, with an affine parameter s, i.e., $\gamma_{q,tX}(s) \equiv \gamma_{q,X}(ts)$. Locally, we may write

$$x := \gamma_{q,X}(s), \qquad y := \gamma_{q,X}(-s), \qquad \text{with Jacobian matrix} \quad \frac{\partial(x, y)}{\partial(q, X)}(s).$$

Then one has the change of variables formula:

$$\int\limits_{M\times M} F(x,y)\;\mathrm{d}\nu(x)\;\mathrm{d}\nu(y) = \int\limits_{M}\int\limits_{T_qM} F(\gamma_{q,X}(\tfrac{1}{2}),\gamma_{q,X}(-\tfrac{1}{2}))\,J(q,X;\tfrac{1}{2})\,\sqrt{g(q)}\;\mathrm{d}X\;\mathrm{d}\nu(q),$$

where we introduce

$$J(q,X;s) := s^{-n}\,\frac{\sqrt{\det g(\gamma_{q,X}(s))}\sqrt{\det g(\gamma_{q,X}(-s))}}{\det g(q)}\,\left|\frac{\partial(x,y)}{\partial(q,X)}\right|(s).$$

This object can be computed from the equations of geodesic deviation [5]. The crucial estimate is

$$J(q,X,\tfrac{1}{2}\hbar) = 1 + O(\hbar^2). \tag{6}$$

In the computation of the Jacobian two types of Jacobi fields intervene, let us call them h and $\tilde{h}$, evaluated in different points. The first is obtained through the variation in the coordinates of the base point x: in components,

$$h^\mu_\nu(q,X,s) = \frac{\partial x^\mu_1(q,X,s)}{\partial q^\nu};$$

the second one using the variation of the tangent vector,

$$\tilde{h}^\mu_\nu(q,X,s) = \frac{\partial x^\mu_1(q,X,s)}{\partial X^\nu};$$

and analogously for x_2 at $-s$. To check the calculations, we may work in normal coordinates, in which

$$h^\mu_\nu = \delta^\mu_\nu(1+O(\hbar^2)), \quad \tilde{h}^\mu_\nu = \hbar\,\delta^\mu_\nu(1+O(\hbar^2)),$$

and moreover the metric has a simple expression.

Connes' quantization/dequantization maps are just the consequence of applying the Gelfand–Naĭmark functor to the tangent groupoid construction and are given simply by

$$Q = \Phi_*^{-1}\circ J^{-1/2}\circ\mathcal{F}^{-1}$$

and:

$$Q^{-1} = \mathcal{F}\circ J^{1/2}\circ\Phi_*\,.$$

Here $J^{-1/2}$, $J^{1/2}$ mean the corresponding multiplication operators. It is possible to rewrite the formulae so J only appears once, but then we would lose property (5). So there is actually a family of "Moyal" quantizations in the general case, unless we demand traciality (as we do). This is a feature of the nonflat case: note that $J = 1$ in the framework of our Section 1. In the main development in [5] J actually only appears in the dequantization formula, but again Landsman gives indication on how to modify the formulae to get traciality. Those factors are analogous to the preexponential factors that appear in the semiclassical expression for the path integral: see, e.g., [16, p. 95]. One can insert them in several ways without altering the axioms, precisely on account of (6).

A bit more explicitly, the previous formulae are

$$\begin{aligned} k_a(x, y; \hbar) &= J^{-1/2}(q, X, \tfrac{1}{2}\hbar)\, \mathcal{F}^{-1} a(q, X), \\ a(q, \xi) &= \mathcal{F}[J^{1/2}(q, X, \tfrac{1}{2}\hbar)\, k_a(x, y; \hbar)], \end{aligned} \tag{7}$$

where (x, y) and (q, X) are related by $x = \gamma_{q,X}(\frac{1}{2}\hbar)$, $y = \gamma_{q,X}(-\frac{1}{2}\hbar)$.

The long but relatively straightforward verifications in [5] then show that we have defined an (obviously real) preasymptotic morphism which moreover is a strict quantization from $C_c^\infty(T^*M)$ to $\mathcal{K}(L^2(M))$. In addition, the tracial property (5) is satisfied. In the Riemannian context we have total tubular neighbourhoods, and it is clear that Fourier transforms of smooth functions of compact support in T^*M decay fast enough that our formulae make sense for them; therefore the quantization is *flabby*.

We make a final comment on the uniqueness, or lack of it, of the quantization considered here. Among our choices are those we make to establish the isomorphism between TM and $\mathcal{N}^\Delta$ on which the whole construction hinges. Regarding only the differential structure, those choices are parametrized by, say, the pair $(\varphi_1 + \varphi_2, \varphi_1 - \varphi_2) \in \mathrm{End}(TM) \times GL(TM)$. However, the identification of the normal bundle with the orthogonal bundle to the diagonal leads to the equation $\varphi_1 + \varphi_2 = 0$, if we take the natural metric on $M \times M$. To the same equation leads the reality constraint (4), as a simple calculation from (7) shows.

After so narrowing the freedom of parameter choice to $GL(TM)$, semitraciality implies at least $\det(\varphi_1 - \varphi_2) = 1$. Connes' condition (2) and our discussion in Section 1 strongly suggest to adopt the restriction $\varphi_1 - \varphi_2 = 1$. Let us agree to call all the quantizations (parametrized by $\mathrm{End}(TM)$) for which the last equation holds quantizations *of the Connes type*. Then we have proved:

Theorem 7 *For any Riemannian manifold, the only* ***real*** *quantization rules of the Connes type are Moyal quantizations.*

The classification of the various quantizations obtained by relaxing the constraints imposed here deserves further investigation.

4 Conclusions and Outlook

The nicest aspect of Connes' construction is perhaps that the continuity requirements in deformation quantization can be given a canonical formulation in C^*-algebraic terms. In fact, we have a continuous field of C^*-algebras in the sense of Dixmier [17], parametrized by $[0, \hbar_0]$. This constitutes a *strong* deformation of $C_0(T^*M)$ into $\mathcal{K}$, i.e., the field of C^*-algebras is trivial except at the origin.

There is a short exact sequence of C^*-algebras

$$0 \longrightarrow C_0(0, \hbar_0] \otimes \mathcal{K} \longrightarrow C^*(G_M) \xrightarrow{\sigma} C_0(T^*M) \longrightarrow 0$$

that yields isomorphisms in K-theory:

$$K_j(C^*(G_M)) \xrightarrow{\sigma_*} K_j(C_0(T^*M)) = K^j(T^*M) \qquad (j = 0, 1).$$

This corresponds to Proposition II.5.5 of Connes [2] and the claims immediately thereafter. It can be seen as follows: given a smooth groupoid $G = G_1 \uplus G_2$ which is a disjoint union of two smooth groupoids with G_1 open and G_2 closed in G, there is a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(G_1) \longrightarrow C^*(G) \xrightarrow{\sigma} C^*(G_2) \longrightarrow 0$$

where σ is the homomorphism of restriction from $C_c^\infty(G)$ to $C_c^\infty(G_2)$: it is enough to notice that σ is continuous for the C^*-norms because one takes the supremum of $\|\pi_u(a)\|$ over the closed subset $u \in G_2^{(2)}$, and it is clear that $\ker\sigma \simeq C^*(G_1)$. Since $C^*(M \times M) = \mathcal{K}$, the C^*-algebra $C^*(G_1)$, obtained by completing the algebraic tensor product $C_c^\infty(0, \hbar_0] \odot C_c^\infty(M \times M)$, is $C_0(0, \hbar_0] \otimes \mathcal{K}$, which is contractible via the homotopy $\alpha_t(f \otimes A) := f(t\,\cdot) \otimes A$ for $f \in C_0(0, \hbar_0]$, $0 \leq t \leq 1$; in particular, $K_j(C_0(0, \hbar_0] \otimes \mathcal{K}) = 0$. At this point, we appeal to the six-term cyclic exact sequence in K-theory of C^*-algebras [18]; the two trivial groups break the circuit and leave the two advertised isomorphisms.

The restriction of elements of $C^*(G)$ to the outer boundary $M \times M \times \{\hbar_0\}$ gives a homomorphism

$$\rho : C^*(G_M) \to C^*(M \times M \times \{\hbar_0\}) \simeq \mathcal{K},$$

and in K-theory this yields a homomorphism $\rho_*: K_0(C^*(G_M)) \to K_0(\mathcal{K}) = \mathbb{Z}$. Finally, we have the composition $\rho_*(\sigma_*)^{-1}: K^0(T^*M) \to \mathbb{Z}$, which is just the Atiyah–Singer analytical index map. (To see that, one must identify suitable pseudodifferential operators with the kernels in $C^*(G_1)$: see [13,19]; of course, general results in the theory of pseudodifferential operators [20] ensure that the index does not depend on the particular quantization recipe chosen.)

To summarize, we found Connes' tangent groupoid construction to address Rieffel's questions concerning the rôle of a Riemannian metric in a strict (deformation) quantization. The presence of a Riemannian metric on a manifold M allows one to identify a geodesic spray to construct a total tubular neighbourhood; to choose a standard representative for the normal bundle, to have a natural isomorphism between the normal and the tangent bundle; to use a canonical measure on M; and to define naturally the fibrewise Fourier transform.

Work on extensions of some of the methods in this paper to more general contexts is in progress.

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