Ah, that 3/8!

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Abstract

The GUT value of Weinberg's angle is also the value that minimises Z^0 decay, independently of any GUT consideration. We review this result and some related facts.

From any textbook (eg [1]), the amplitude for decay of Z^0 into a fermion pair, at leading order, is

$$\Gamma(Z^0 \to f\bar{f}) = C_f(|V_f|^2 + |A_f|^2)\frac{G_F M_Z^3}{6\sqrt{2}\pi}$$

where C_f is a color normalisation constant, 1 for fermions and 3 for quarks, and V_f and A_f are the vector and axial charges,

$$\begin{aligned} V_f &= T_f^3 - 2Q_f \sin^2\theta_W \\ A_f &= T_f^3 \end{aligned}$$

If we are interested on the decay into a set of fermions, we add the contributions:

$$K_{\{f\}} = \sum_f C_f((T_f^3)^2 + (T_f^3 - 2Q_f\hat{s})^2)$$

We want to know for which value of $\hat{s} \equiv \sin^2\theta_W$ will the relative coupling, and then the decay width, to be a minimum. Thus we ask

$$0 = K'_{\{f\}}(\hat{s}) = \sum_f 2C_f(T_f^3 - 2Q_f\hat{s})(-2Q_f) = 4\sum_f C_f(2Q_f^2 - T_f^3 Q_f)$$

and then using that $T_f^3 Q_f = T_f^3(Y + T_f^3) = (T_f^3)^2$, accounting colour in the sum, and passing the sum from Dirac to Weyl species we get

$$\hat{s}_{min} = \frac{\sum_f T_f^3 Q_f}{2\sum_f Q_f^2} = \frac{\sum (T_f^3)^2}{\sum Q_f^2}$$

When the set of fermions is a whole generation, this last formula equals the very well known result (e.g. exercise VII.5.2 in [2]) for $\sin^2\theta_W$ at the GUT scale of any unification based on a simple group. It is independent of the specific fermion content of the theory.

In particular for the fermion content of a generation of the standard model, we have $\hat{s}_{min} = 3/8$ (and $K_{\{u,d,\nu,e\}}(3/8) = 2.5$).

The charge assignments of the standard model can be imposed by hand or via the requisites of anomaly cancellation. In any case, they have an extra property

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when we pay attention to minimisation of Z^0 decay: the value 3/8 also minimises separately the partial decay width towards an u (or c) quark. And thus it minimises also the partial decay into the set $\{d, \nu_e, e\}$ (or $\{s, \nu_\mu, \mu\}$ or $\{b, \nu_\tau, \tau\}$)

This means that for the standard model, 3/8 is not only the value minimising decay into first and second generation; it also minimises the decay of Z^0 into the third generation, even if the top quark is kinematically out of reach of the gauge meson. A posteriori, we could interpret this fact of an indication of the particular characteristics of the top quark.

Up to here the main comment, or result[1], of this note: that the GUT formula for Weinberg angle is also got without GUT, by minimising Z_0 decay. The following few paragraphs are random musings distilled from the above:

- A consequence of the derivation here presented is that a model can get into GUT angle by asking for some minimisation requisite, without looking for a GUT group. Spectral actions of Connes-Chamseddine could be a good candidate for this, and I wonder if Ibañez string-inspired approaches to Weinberg angle are also a consequence of hidden minimisation.

- If we contemplate $K_{\{u,d,\nu,e\}}$ we can wonder for the value of this coupling at the experimental scale of decay, ie when $\hat{s}$ is about 0.232. An unexplained fact is that

$$K_{\{u,d,\nu,e\}}(0.231948...) = \exp(1) = \sum_0^\infty \frac{1}{n!}$$

We haven't the slightest idea of why the transcendent number e could have a reason to appear here. The minimum, $K = 2.5$, is a member of the simple series expansion of e, up to three terms. But on the other hand the values $K = 1$ and $K = 2$, which we could get by using the lower terms, need of a complex $\hat{s}$.

Also for the standard model assignment of charges, and in terms of the decay to a whole family, we have the relationships $K_{\{d,e\}} = \frac{1}{2}K$, $K_\nu = \frac{1}{2}$, $K_u = \frac{1}{2}K - \frac{1}{2}$. This implies that for the above mentioned $K = 1$, the decay probability into upper quarks vanishes.

- Just for analytic commodity, we can solve the equation for $\hat{s}$ in terms of the decay to a whole family. We have

$$\hat{s} = \frac{3}{8}(1 - \sqrt{\frac{2}{3}}\sqrt{K - \frac{5}{2}})$$

- Finally, let me note that another common aparition of the factor 3/8 is in perturbative expansions of electromagnetism, and that the use of the GUT to correct α_{EM} has been vindicated in some exponential adjustments between Planck and electron scales, eg by Laurent Nottale. I strongly doubt that the development here can be connected to these ones.

References

[1] R. K. Ellis, W. J. Stirling and B. R. Webber, "QCD and collider physics," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **8**, 1 (1996).

[2] A. Zee, "Quantum field theory in a nutshell," Princeton University Press (2003).

[1] The whole note is motivated because I have been unable to find this remark in standard textbooks; I'd thank any information about previous statements of it