Discrete spectral triples converging to dirac operators.

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Abstract

We exhibit a series of discrete spectral triples converging to the canonical spectral triple of a finite dimensional manifold. Thus we bypass the non-go theorem of Gökeler and Schücker. Sort of counterexample.

In [2] Connes gave the most general example of non commutative manifolds defined from finite spectral triples, and such discrete manifolds were classified independently by Krajewsky [6, 7] and Paschke and Sitarz [8] the same year. But almost the same time, Gökeler and Schücker [3] proved that the naive discretizations of Dirac Operators were not able to fulfil the axioms of non-commutative manifolds. This caused a fading in the attempts to illuminate NCG from the point of view of lattice theories. Later attempts to show NCG operations in lattices have preferred to avoid the full axiomatic required by K-theory and Reality [1]. But the subsequent works on lattices, from the successful show of Lüscher with the index theorem, to the insistent studies of Jian Dai or M. Requardt, to name some ones, should invite us to try to develop the full geometric set-up.

In [3], the non-go principles also ruled out some non-naive formulations. At that time, inspired on some previous speculations, I risked to suggest privately the use of a doubling of the representation of the algebra, thus a function $A(x)$, x in one dimension, could be discretized to an operator $A(i)$ with representation:

$$A|v>=\begin{pmatrix} \ddots & & & & & \\ & A_{i-2} & & & & \\ & & A_{i-1} & & & \\ & & & A_{i-1} & & \\ & & & & A_i & \\ & & & & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ v_{i-1}^- \\ v_{i-1}^+ \\ v_i^+ \\ v_i^- \\ \vdots \end{pmatrix}$$

Regretfully this doubling is not a non-commutative manifold, as a check of [6] or [8] can show. So it was not a good idea. Now, it happens that the axioms can be

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[1] In NCG, Reality refers to Atiyah' work. I can not resist to quote J.P. May: "Atiyah introduced Real K-theory KR, which must not be confused with real K-theory KO. In the paper, real vector bundles mean one thing over *real spaces* and another thing over *spaces*, which has bedeviled readers ever since: we distinguish Real from real, never starting a sentence with either". It is a pity that Atiyah's original does not follow this suggestion.

met if we opt by a tripling, i.e. a representation of $A(i)$ as:

$$A|v>=\begin{pmatrix} \ddots & & & & \\ & A_{i-1} & & & \\ & & A_i & & \\ & & & A_{i+1} & \\ & & & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ v_i^- \\ v_i^0 \\ v_i^+ \\ \vdots \end{pmatrix}$$

This representation corresponds, in Connes example, to the commutative discrete spectral triple with intersection matrix[2]:

$$q=\begin{pmatrix} -1 & 1 & & 0 & b \\ 1 & -1 & 1 & & 0 \\ & 1 & \ddots & 1 & \\ 0 & & 1 & -1 & 1 \\ b & 0 & & 1 & -1 \end{pmatrix}$$

where $b=1$ to get a discrete circle, and $b=0$ to get a segment.

There were in principle two reasons to disregard this triple. On one hand, the intersection matrix seems to be degenerated. It is not. On the other hand, the 3x3 matrix over each point seems not to be able to reproduce a set of Dirac (or at least Pauli) matrices in the limit. It can.

The degeneracy of the matrix is accidental and depends of the dimension of the matrix, this is, of the number of points we are using in the corresponding discrete lattice. We can take determinant of q and we see[3] that the sequence is, for the circle,

$$1,-3,4,-3,1,0,1,-3,4,-3,1,0,1,-3,\ldots$$

where the first 0 happens for n=6 and with periodicity six, while for the the segment it happens first for n=2 and then with periodicity 3:

$$-,0,1,-1,0,1,-1,0,1,-1,0,1,\ldots$$

so in both cases is it possible to take a non degenerate subsequence. Moreover, one can cast some doubt about the causes of the failure of the intersection form: the axiomatic asks for Poincare duality in the K-theory, and additional properties must be requested to the Chern character in order to get the intersection form as commonly is calculated. Perhaps a fine-tuning in the Dirac operator could give us a character wiping away the degeneracy of the intersection.

As for the second point, one can observe that the operator $[D,A]$ has always a null space of dimension one third of the total. In the infinite limit, this nullspace becomes independent of A and we recover the usual expression $i\sigma_1\partial_x$.

Lets build more explicitly the spectral triple, following the instructions of [2]. The hilbert space is the sum of subspaces $\oplus H_{ij}$, the dimension of each subespace being given by the absolute value of the element q_{ij}. There is a chirality operator γ that acts on H_{ij} as ± 1, following the sign of q_{ij}. There is also a involution J with action $J|e_{ij}>=|\bar{e}_{ji}>$

The action of A in each H_{ij} is multiplication times $A(i)$. This is the kind of matrix mentioned above. Note that JAJ^{-1} builds the opposite algebra A^O.

Finally, the dirac operator is given by elements $m_{ij,kl}$ having some restrictions: first, there are different of zero only if $i=k$ or $j=l$. Second, they are different of

[2]I want to thank the colective aid of the people of usenet newsgroup `sci.math`, time ago, to understand this family of matrices

[3]Of course there are two null eigenvectors for the circle but only one for the segment

zero only when the pairs ij and kl show different sign (ie chirality) in the matrix q. Third, they are restricted by the symmetries $m_{ij,kl} = \bar{m}_{kl,ij}$ and $m_{ij,kl} = \bar{m}_{ji,lk}$. With these symmetries, is is easy to check that that the differential [D,A] is a matrix of diagonal boxes, acting in the subespaces $H_{-,l}$ and codifying the forward and backward derivatives, a'_-, a'_+ at the point l.

It is instructive to see the nullspace of the differential with some detail. Thus explicitly, for a given point l, the operator $[D, A]$ on the subspace $H_{l-1,l} \oplus H_{l,l} \oplus H_{l+1,l}$ is:

$$[D, A]| >= \begin{pmatrix} 0 & (a_l - a_{l-1})m_{l-1\,l,ll} & 0 \\ (a_{l-1} - a_l)m^*_{l-1\,l,ll} & 0 & (a_{l+1} - a_l)m_{ll,l+1\,l} \\ 0 & (a_l - a_{l+1})m^*_{ll,l+1\,l} & 0 \end{pmatrix} \begin{pmatrix} e_{l-1,l} \\ e_{ll} \\ e_{l+1\,l} \end{pmatrix} =$$

$$= i \begin{pmatrix} 0 & a'_- & 0 \\ a'_- & 0 & a'_+ \\ 0 & a'_+ & 0 \end{pmatrix} \begin{pmatrix} e_{l-1,l} \\ e_{ll} \\ e_{l+1\,l} \end{pmatrix}$$

(note that A^o acts here diagonnally, $A(l)\mathbf{1}_3$, fullfilling the first order condition of the differential calculus)

The nullvector comes from a lineal combination of the forward and backward derivatives, $|\Omega >= a'_+ |e_{l-1,l} - a'_- |e_{l+1,l} >$. We can rotate this vector to the basis to see explicitly the zero; get vectors $(0, \sqrt{a'^2_- + a'^2_+}, 0)$ and $(-a'_-, 0, a'_+)$ and then $[D, A]$ is:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\sqrt{(a'_-)^2 + (a'_+)^2} \\ 0 & i\sqrt{(a'_-)^2 + (a'_+)^2} & 0 \end{pmatrix} \begin{pmatrix} e_\alpha \\ e_{ll} \\ e_\beta \end{pmatrix}$$

In the limit, forward and backward derivatives always coincide, so the (normalized) change is the same for each box and can be done simultaneously over the total summed vectors

$$E^- = \oplus_l e_{l-1,l},\ E^+ = \oplus_l e_{l+1,l},\ E^0 = \oplus_l e_{l,l}$$

The nullspace $E^- - E^+$ is completely contained in the positive chirality sector; which cures a non-problem; we had built a hilbert space where the positive chirality had double number of degrees of freedom that the negative one. In the commutative limit, bot chiralities are equal. The chirality operator, γ, is then other Pauli matrix.

So we see that it is possible to build a one-dimensional manifold as the limit of zero dimensional spectral triples. The discrete spectral triple is 0-dimensional. So we can apply the rule for the product of even-dimensional spectral triples to get 0-dimensional triples having as limit higher dimensional manifolds. Just to check as it works, we can multiply the basic spectral triple by itself. The algebra is now the direct product of two copies of A, and the same rule applies to the hilbert spaces H and, at least in the even case, to the operator J. As for the dirac operator, in this case (and to get a good rule for D^2) it is defined to be:

$$D_2 = D \otimes 1 + \gamma \otimes D$$

So the commutator $[D_2, A^x \otimes A^y]$ results

$$[D, A^x] \otimes A^y + \gamma A^x \otimes [D, A^y]$$

getting a new Pauli matrix in the second term; the continuous limit give us the previous σ_1 and then the result is

$$\lim[D_2, A \otimes A] = i\sigma_1 \partial_x A^x \otimes A^y + (-\sigma_3)A^x \otimes i\sigma_1 \partial_y A^y \equiv i(\sigma_x \partial_x + \sigma_y \partial_y)$$

not so good as expected.

It is possible to iterate this process, first doing the calculation in the discrete, then taking the limit. As the products are always of 0-dimensional spectral triples, this method avoids the doubts[9] about which sign rule applies to the product, depending on dimension.

To end, some remarks:

a) It still seems interesting to study the action $Tr_\omega(D^{n-2})$, where n is the dimension of the triple.

b) The naive attempt, with only a doubling, merits further study. It is apparent that the q matrix is invertible or degenerate depending of the existence of boundary for the limit manifold. Also it needs of an even number of points, to hide the inhomogeneity of this kind of spaces of alternating chirality. In the lattice, should be interesting a connection to continuum limits of antiferromagnetic states.

c) The motivation to try this discretization comes from previous conjectures about physicsl calculus. But while physically it could exist some suggestion, mathematically one feels unsure about the need of this duplication (and triplication) of degrees of freedom. Besides the usual stuff on Clifford Algebras, I wonder if it could be related with the complications to embbed a manifold in an euclidean space... it could be a motivation to go to the cinema this month. But if it were, it is good to remember [1] that the non-riemannian metrics need higher dimensions, going (casually) up to dimension ninety for the (3,1) metric[4].

d) It seems that there is some independence between the forward and backward derivatives, box-to-box. If this can be manifest and if this implies some freedom in D, which could survive as discrete space in the continous limit, it remains as a note for further research.

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[4] To be precise, $2+46$ for compact spaces and $2+87$ for the non-compact case. But see also [4]